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COMPARISON OF WATER SUPPLY SYSTEMS FROM A FINANCIAL POINT OF VIEW.

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WITH DISCUSSION.

The tables given in the following brief paper were not collected by the writer for the purpose of presentation to this Society, but in order to prove on general principles certain assumptions in relation to the comparative efficiency of the three general systems of water supply. The data was gradually extended as each new fact failed in its turn to perform the expected duty, until finally was brought about, if not a change of view, at least an agnostic position in the mind of the writer.

Although there are probably a number of misstatements in the tables, yet a sufficiently large number being verified by comparison with official reports, due allowance being made for differences in dates, it is believed that they are, in the main, correct.

For the data by means of which Table No. 3 and the greater part of Table No. 1 are calculated, credit must be given to the invaluable "Manual of American Water-Works for 1888."

The thanks of the writer are due to the superintendents of the

TABLE No. 1.

Name of Town.	Date of First Construction.	Estimated Population.	Miles of Mains.	Number of Taps.	Total cost divided by 3 650.	Annual operating expenses divided by 365.	Pressure in pounds.	Gallons consumed daily in millions.	Foot pounds in millions.	Foot pounds in millions for each $1 of total cost, divided by 3 650.	Foot pounds in millions for each $1 of daily expense.	Foot pounds in millions for the sum of the preceding.	Foot pounds in millions for total daily expenditures considered on the basis of 10 lbs. pressure.	Minimum meter rates in cents.	Total Cost for 1 000 gallons in cents.
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
SYSTEM.—PUMPING TO RESERVOIR.															
Cincinnati, O.	1820	275 000	247	31 128	$2 158	$752	100	32 079	26 626	12.3	35.4	9.1	0.91	9	9.1
Baltimore, Md.	1807	400 000	358	66 089	2 994	293	60	34 000	16 932	5.6	57.8	5.1	0.85	8	9.6
Pittsburgh, Pa.	1826	215 000	175	16 000	1 239	545	70	30 000	17 430	14.1	31.9	9.8	1.40	5	5.9
Brooklyn, N. Y.	1856	770 000	397	79 073	3 693	1 603	40	46 278	15 364	4.3	9.5	2.9	0.72	10½	11.2
Philadelphia, Pa.	1800	1 000 000	876	163 816	4 125	1 624	20	88 850	14 749	3 5	9.1	2.5	1.25	8	6.4
Boston, Mass.	1848	400 000	574.3	72 039	5 826	1 362	38	37 481	11 827	2.0	8.7	1.6	0.42	20	19.2
St. Louis, Mo.	1830	450 000	315	34 022	3 575	743	40	31 500	10 458	2.9	14.0	2.4	0.60	12½	13.6
San Francisco, Cal.	1856	250 000	250	26 000	4 675	2 200	60	19 112	9 536	2.0	4.3	1.4	0.23	.....	36.0
Detroit, Mich.	1836	200 000	323	32 938	1 125	294	25	26 079	7 487	6.6	25.4	5.2	2.08	10	9.9
Cleveland, O.	1854	230 000	252	22 658	1 490	247	40	22 266	7 392	5.0	30.0	4.4	1.10	6	7.8
Milwaukee, Wis.	1872	185 000	148	14 072		329	38	20 089	6 336	7.6	19.2	5 4	1.42	4	5.7
Allegheny, Pa.	1847	80 000	27	3 300	459	248	40	17 824	5 918	13 1	23.8	8.5	2.12	15	3.9
Omaha, Neb.	1880	100 000	75	3 500	825	124	100	7 500	6 225	7.5	50.2	6.5	0.65	.....	12.6
Jersey City, N. J.	1852	155 000	160	16 936	1 362	1 482	40	16 500	5 478	4.0	3.7	1.9	0.47	16⅔	17.2
Richmond, Va.	1830	85 000	73.4	8 900	413	84	41	11 133	3 788	9.1	45.1	7.6	1.85	.....	4.4
Harrisburg, Pa.	1844	40 000	30	6 000+	192	68	60	6 400	3 187	16.6	46.8	12.2	2.03	.....	4.0
Louisville, Ky.	1857	165 000	137.4	11 001	1 378	163	35	10 307	2 904	2.2	18.3	1.9	0.54	6	15.0
Portland, Ore.	1858	40 000	18	.....	138	130	60	5 000	2 490	18.0	19.1	9.3	1.55	.....	5.3
Nashville, Tenn.	1832	80 000	42	4 321	550	119	35	8 523	2 476	4.5	20.8	3.6	1.02	.....	7.8
Newark, N. J.	1800	160 000	161	12 207	734	284	23	12 642	2 413	3.3	8 5	2.3	1.00	5	8.1
Lowell, Mass.	1872	70 000	83	7 440	1 174	92	60	4 319	2 151	1.8	23.3	1.6	0.27	15	29.4
Leavenworth, Kan.	1882	31 000	23	1 128	103	41	125	1 800	1 867	18.1	45.5	12.9	1.03	.....	8.0
Cohoes, N. Y.	1848	21 000	25	1 200	73	33	70	3 000	1 743	23.8	52.9	16.4	2.34	.....	2.5
Wheeling, W. Va.	1831	35 000	41	6 000	206	76	45	4 500	1 691	8.1	22.1	5.9	1.31	.....	6.2
Erie, Pa.	1887	25 000	56.5	3 225	2225	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....

Erie, Pa.	1867	35 000	56.5	6 367	$268.	$48	60	3 335	1 661	6.2	34.6	5.2	.87	6	9.6
Providence, R. I.	1870	125 000	202	13 128	1 714	192	39	4 925	1 554	0.9	8.1	0.8	.20	15	38.9
Toledo, O.	1873	80 000	56	2 875	344	63	50	3 700	1 536	4.4	24.3	3.7	.74	8	11.0
Wilmington, Del.	1804	42 000	69.5	10 869	209	138	35	5 272	1 532	7.3	11.1	4.4	1.25	10	6.5
Grand Rapids, Mich.	1873	60 000	28	1 750	146	62	65	2 769	1 498	10.2	24.1	7.2	1.10	.....	7.3
Sacramento, Cal.	1854	35 000	35	4 600	96	69	30	6 000	1 494	15.5	21.6	9.0	3.00	.....	2.7
Lawrence, Mass.	1873	40 000	53.6	4 447	537	64	70	2 500	1 453	2.7	22.7	2.4	.34	20	24.0
Hoboken, N. J.	1855	75 000	78.5	5 559	268	149	40	4 250	1 411	5.2	9.3	3.4	.85	.....	9.7
Oswego, N. Y.	1868	24 000	38	1 000	138	45	50	3 000	1 245	9.0	27.6	6.7	1.34	.....	6.1
New Orleans, La.	1833	230 000	72	4 400	659	109	25	6 000	1 245	1.9	11.4	1.6	.64	.....	12.8
Lynn, Mass.	1870	51 400	81	7 800	377	61	62	2 379	1 225	3.2	20.0	2.8	.45	20	18.2
Little Rock, Ark.	1878	35 000	12	1 044	118	65	70	2 400	1 162	9.9	46.4	8.1	1.15	.....	9.1
Council Bluffs, Ia.	1882	35 000	25	1 400	172	28	82	1 700	1 157	6.7	41.3	5.8	.71	.....	11.7
San Antonio, Tex.	1878	45 000	70	4 000	187	55	65	2 000	1 079	5.7	19.6	4.4	.67	.....	12.1
Fall River, Mass.	1871	65 000	60	4 197	482	69	80	1 591	1 056	2.2	15 3	1.9	.24	20	34.4
Springfield, O.	1881	35 000	32.5	1 699	120	36	80	1 500	996	8.3	27.6	6.3	.78	20	10.4
Lancaster, Pa.	1836	32 000	32	5 200	165	32	28	4 000	930	5.6	29.1	4.7	1.68	.....	4.9
Sandusky, O.	1875	23 000	25	1 618	113	35	60	1 759	876	7.7	25.0	5.9	.98	20	8.2
Manchester, N. H.	1872	41 000	49.5	3 100	248	41	62	1 750	901	3.6	22.0	3.1	.50	26⅔	16.0
Camden, N. J.	1860	55 000	52.7	908	192	55	20	5 267	874	4.5	15.9	3.5	1.75	.....	4.6
Salem, Mass.	1865	28 000	50	5 016	392	70	40	2 580	856	2.2	12.2	1.8	.45	.....	17.7
New Bedford, Mass.	1865	36 000	54.4	5 495	357	72	33	3 047	835	2.3	11.5	1.9	.57	.....	14.3
Elizabeth, N. J.	1854	35 000	45	3 000	275	110	30	2 500	623	2.2	5.6	1.6	.53	.....	15.4
Lynchburg, Va.	1828	25 000	16	1 300	47	22	60	1 250	622	13.2	28.3	9.0	1.50	.....	5.3
New Port, Ky.	1872	27 500	26.6	3 256	209	27	80	900	598	2.8	22.1	2.5	.31	15	26.2
Poughkeepsie, N. Y.	1870	22 000	18.4	1 790	161	50	40	1 767	587	3.6	11.7	2.8	.70	.....	11.7
Trenton, N. J.	1855	40 000	40	8 102	123	41	30	2 256	562	4.5	13.7	3.4	1.13	.....	7.1
Springfield, Ill.	1866	30 000	32	1 700	151	61	20	2 441	405	2.6	6.6	1.9	.95	.....	8.8
Quincy, Ill.	1873	36 000	20	1 150	12	41	35	671	195	16.2	4.7	3.6	1.02	15	7.9
Petersburg, Va.	1856	25 000	11	1 080	42	25	24	750	149	3.5	6.0	2.2	0.91	.....	9.0
								Average	.....	6.85	22.30	4.85	1.01	.....	11.77

SYSTEM.—DIRECT PUMPING.

Chicago, Ill.	1852	775 000	638.2	100 000	$3 132	$2 794	25	91 648	19 017	6.0	6.8	3.2	1.28	8	6.4
Buffalo, N. Y.	1852	250 000	250	29 467	1 238	349	30	38 485	9 583	7.7	27.4	6.0	2.00	3	4.2
Kansas City, Mo.	1873	110 000	85	7 786	1 238	154	90	10 000	7 470	6.0	48.5	5.4	.60	15	13.9
Minneapolis, Minn.	1867	160 000	112	5 598	391	187	80	10 814	7 181	18.3	38.4	12.4	1.55	10	5.2
Columbus, O.	1870	95 000	70.2	5 207	319	84	70	5 816	3 379	10.5	40.2	8.3	1.18	7	6.9
Savannah, Ga.	1853	52 000	32.6	2 850	83	36	30	5 715	1 423	17.1	39.5	11 9	3.96	.....	2.1
Des Moines, Ia.	1871	45 000	40	2 500	275	55	80	2 000	1 328	4.8	24.1	4.0	0.50	.....	16.5
Evansville, Ind.	1870	60 000	33.5	2 230	138	109	40	4 000	1 328	9.6	12.2	5.3	1.32	.....	6.2
Schenectady, N. Y.	1871	25 000	17	1 300	41	21	85	1 500	1 058	25.8	50.4	17.0	2.00	8	6.2
Davenport, Ia.	1874	31 000	28	1 523	237	69	60	2 000	996	4.2	14.4	3.2	.53	.....	15.3
Peoria, Ill.	1867	45 000	47	3 000	124	88	40	3 000	996	8.0	11.3	4.6	1.15	.....	7.1
East Saginaw, Mich.	1873	32 500	40	1 911	134	44	40	2 813	934	7.0	21.2	5.2	1.30	.....	6.3

TABLE No. 1.—*Continued.*

Name of Town.	Date of first Construction.	Estimated Population.	Miles of Main.	Number of Taps.	Total cost divided by 3 650.	Annual operating expenses divided by 365.	Pressure in pounds.	Gallons consumed daily in millions.	Foot pounds in millions.	Foot pounds in millions for each $1 of total cost, divided by 3 650.	Foot pounds in millions for each $1 of daily expense.	Foot pounds in millions for the sum of the preceding.	Foot pounds in millions for total daily expenditures considered on the basis of 10 lbs. pressure.	Minimum meter rates in cents.	Total cost for 1 000 gallons in cents.
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
SYSTEM.—DIRECT PUMPING.—*Continued.*															
Augusta, Ga	1859	46 000	30	1 728	$55	$15	35	2 755	800	14.5	53.3	11.4	3.25	.....	2.7
Dayton, O	1869	55 000	37	1 799	270	52	50	1 851	768	2.8	14.7	2.4	.48	10	17.9
Bay City, Mich	1872	33 000	25.5	1 332	118	32	40	2 266	752	6.3	23.5	5.0	1.25	.....	6.5
Burlington, Ia	1877	30 000	19.5	1 477	55	41	90	1 000	747	13.5	18.2	7.7	.85	.....	9.6
Topeka, Kansas	1881	40 000	28	1 450	55	39	110	800	730	13.2	18.7	7.8	.71	.....	11.7
Norfolk, Va	.....	30 000	30	3 925	218	41	30	2 378	592	2.7	14.4	2.2	.73	.....	10.8
Atlanta, Ga	1874	66 000	27	2 000	124	49	40	1 575	523	4.2	10.6	3.0	.75	.....	10.8
Ft. Wayne, Ind	1880	35 000	30	1 669	86	25	40	1 400	465	5.4	18.6	4.2	1.05	.....	8.0
Terra Haute, Ind	1873	40 000	28	800	95	22	35	1 250	363	3.8	16.5	3.1	.88	.....	9.0
Covington, Ky	1868	38 000	28	2 500	110	60	30	1 500	374	3.4	5.4	2.1	.70	15	12.0
Taunton, Mass	1876	25 000	54.6	2 892	151	44	40	725	241	1.6	5.5	1.2	.30	.....	26.7
								Average	.....	8.54	23.21	5.94	1.23	.....	9.6
SYSTEM.—GRAVITY.															
New York, N. Y	1835	1 700 000	620	180 000	$10 970	$1 644	10	110 000	9 130	0.83	5.56	0.73	.73	13½	11.5
Washington, D. C	1853	160 000	128	22 095	2 338	38	20	26 148	4 340	1.9	114.2	1.83	.91	10	9.1
Troy, N. Y	1880	60 000	53.2	5 250	322	140	60	8 000	3 984	12.4	28.4	8.6	1.43	10	5.8
Worcester, Mass	1852	79 500	110	8 110	482	259	100	4 000	3 320	6.8	12.8	4.4	.44	15	18.5
Springfield, Mass	1873	40 000	81.5	4 819	367	69	90	4 000	2 988	8.1	43.3	6.8	.75	10	10.9
Hartford, Conn	1854	48 000	80	5 836	467	78	60	5 000	2 490	5.3	32.0	4.5	.75	10	10.9
Portland, Me	1867	34 000	96	6 425	413	40	60	5 000	2 490	6.0	62.2	5.5	.92	.....	9.1
Rochester, N. Y	1872	130 000	216	17 000	1 063	83	40	7 200	2 390	2.2	28.8	2.1	.52	5	16.0
Oakland, Cal	1866	45 000	164	7 400	1 430	330	40	7 000	2 324	1.6	7.0	1.3	.32	.....	25.1
Reading, Pa	1865	60 000	58.9	21 940	320	40	50	4 000	1 660	5.2	41.5	4.6	.92	10½	9.0
Holyoke, Mass	1873	31 500	42.4	2 713	132	22	72	2 246	1 341	10.1	60.9	8.6	1.19	.....	6.8
Elmira, N. Y	1860	27 000	40	1 500	143	36	44	2 250	822	5.7	22.8	4.6	1.04	10	7.9
Altoona, Pa	1871	30 000	35.8	5 272	98	44	50	175	73	0.74	1.6	0.51	.10	.....	81.1
								Average	.....	5.14	35.46	4.16	.77	.....	17.0

various water-works whose annual reports furnished the data on which Table No. 2 is based.

Table No. 1 contains water-works data of all towns in the United States having in 1880 a population of over 15 000, and whose reports were sufficiently full for purposes of comparison. The omissions are chiefly among those supplied by private companies. Column 1 contains the names of the towns; column 2, the date of original construction, when the present works are a direct outgrowth of the first, or of the date of complete reconstruction; column 3, the estimated present population, probably in all cases somewhat in excess of the true; column 4, the number of miles of pipes; and column 5, the number of taps. In column 6, 10 per cent. of the total cost is taken as a basis of comparison. Although this may seem excessive, yet when it is remembered that nearly every water-works in the country is practically reconstructed every twenty or thirty years, that the rate of interest paid is from 3 to 7 per cent., and that the life of pipe systems, pumps and boilers, is certainly less than twenty years, it will be seen that the proportion taken is, if anything, too small. Even a gravity system with masonry conduits, intended to last indefinitely, will begin to grow so inadequate, as to involve changes of such a general character, that a much less enduring original construction might well have been adopted had the sequel been known in advance.

Column 7 gives the daily operating expense; column 8, the pressure in pounds; and column 9, the daily consumption in millions of gallons. Column 10 gives the work done in furnishing the supply in millions of foot-pounds. There is evidently no question, when comparing the efficiency of systems, as to the fairness of making the pressure a factor. The lack of it often makes the difference between a small fire and a great conflagration, while an excessively low head for daily service causes (as in New York) an individual expense whose aggregate, although unknown, must be enormous.

Column 11 gives the work done in millions of foot-pounds for each dollar of the 10 per cent. interest and sinking fund; column 12, the same for operating expenses alone; column 13, for each dollar of the total daily expenditure, including both interest and maintenance; column 14 is the same as 13, but with the pressure factor eliminated; column 16 gives the total cost per thousand gallons in cents; it is, therefore, the same as 14, but with the results presented in a different

manner to correspond with common usage. Column 15 gives the minimum meter rates so far as they could be obtained.

It is curious to observe that, if the basis of the table is correct, and the general rates are based on the meter rates, nearly all the water-works in the country sell below the cost of production. Meter rates seem, in fact, when on any basis, to take into account simply the operating expense and annual interest. This is very apparent in Table No. 2.

It is difficult to select any single factor on which the efficiency, or what is the same thing, the proportionate cost, of any one plant depends. The writer plotted the data in nearly every combination, but while there was a tendency toward a law, it was by no means a simple one. Plotted with respect to date of construction, gravity and reservoir systems show a decided tendency to vary inversely with the age of the works, while direct pumping gives a horizontal line. When five-year averages of the two former are taken, not only is the age factor more apparent, but the curves of the two systems are almost exactly parallel between 1855 and 1888. This is the more noticeable from the great irregularities exhibited in certain years, and can therefore be attributed entirely to current market prices.

When plotted with respect to number of miles of mains, direct pumping shows no relation whatever; gravity a slight inverse ratio, that might be accidental; and reservoir systems an inverse ratio in the small systems, but none in the large. It is to be inferred therefore that the size of the town, as indicated by the number of miles of mains, either has no effect on the efficiency, or that its effect is concealed under more important factors.

Plotted with reference to first cost, the total efficiency of gravity and reservoir systems has an inverse ratio, but direct pumping shows no relation.

With reference to daily consumption, the total efficiency of gravity and reservoir systems has a decided inverse, while direct pumping has an equally decided direct ratio to the amount of water used.

Comparing the averages of all, with respect to efficiency based on first cost alone:

Gravity gives	5.14
Pumping to reservoir	6.85
Direct pumping	8.54

If one town (Altoona) be omitted from the list of gravity systems, the efficiency of the latter will be about the same as the reservoir systems.

Based on operating expense alone:

Gravity gives	35.46
Pumping to reservoir	22.30
Direct pumping	23.21

It will be seen, by comparison with Table No. 2, that the difference in favor of the gravity systems represents pump-house expenses, the latter being about half the total for maintenance.

Based on the combined expense of interest, sinking fund, and maintenance:

Gravity gives	4.16
Pumping to reservoir	4.85
Direct pumping	5.94

Correcting the gravity average for the one town will make it nearly the same as the reservoir systems, leaving a balance in favor of direct pumping of about 20 per cent.

Taken on the same basis as the preceding, but with the pressure reduced to 10 pounds:

Gravity gives	0.77
Pumping to reservoir	1.01
Direct pumping	1.23

The percentage in favor of direct pumping being the same as the preceding.

The total average cost per thousand gallons in cents will be:

Gravity	17.1
Pumping to reservoir	11.77
Direct pumping	9.6

Again omitting Altoona from the gravity systems, the latter will give an average of twelve cents per thousand gallons. Direct pumping as before has 20 per cent. advantage.

From this table we are, therefore, justified in concluding that the efficiency for large towns is the same for gravity and reservoir systems, and that direct pumping has about 20 per cent. advantage; that for the former two it will vary inversely with the age of the works, the size of the town, the quantity of water consumed, and the original cost. Also, that the efficiency of direct pumping in the works reported is independent both of the age and original cost, and increases directly with the consumption. These ratios do not, of course, apply to small towns where the addition of a reservoir of twenty hours capacity adds but 12 per cent. to the total cost and diminishes the pumping expenses by nearly one-half.

Table No. 2 is deduced from current official reports of the towns in question, and is self-explanatory. Pressures are given in feet, the dynamic head being used when procurable. The interest is that actually paid by the town, and includes no sinking fund account. The prices are unduly high, compared with Table No. 1, being compiled chiefly from New England reports. Taking them as they are, the percentage to be added to the total cost, in order to make the price per thousand gallons cover the sinking fund used in Table No. 1, is for gravity, 70; for reservoir, 30; and for direct pumping, 110 per cent.

It will be noticed that the cost per thousand gallons due to the interest account is for gravity, 67; for reservoir, 63; and for direct pumping, 51 per cent. of the total. The total operating expense is nearly the same for gravity and direct pumping, the difference in favor of the latter being due to a smaller first cost. The operating expense for gravity and reservoir systems is the same, if from the latter be deducted the pump-house account.

The number of examples taken is too small to show the ratios of Table No. 1.

Table No. 3 is added to show what has been the practice of designers of water-works in this country. In the percentage column it will be seen that the proportion of gravity systems grows smaller with the date. This is due to several reasons, among the most important of which are: All cheaply available supplies will be secured as soon as the need is felt, so that a great number will date almost from the beginning of the history of the town; favored localities are comparatively few in number; and, finally, the total proportion is greatly dwarfed owing to the number of smaller works built during recent years. On the other hand, the ratio is increased beyond its merits, owing to a popular prejudice that compels a designer, whatever his private judgment may be, to seek a gravity supply whose possible cost will be based less on its relative value than on the paying ability of the town.

Reservoir systems keep nearly the same ratio throughout.

Direct pumping came into favor about 1870. It was at first regarded more as a makeshift, compelled by the combination of general financial depression and high prices of material, than as a permanent system, but as pumps more adapted to that special service appeared, the method soon grew into favor as a recognized system of supply. Being under great disadvantage when the consumption is small, the reservoir-stand-

Name of Town.	Kind of System.	Head in feet.	Quantity in Million Gallons.	Date of Report.	Cost of Water per 1 000 Gallons in Cents, Due to:				Total cost per 1 000 Gallons in Cents.	Meter Rate in Cents.
					Interest on Bonds.	Engineering Administration or Superintendence.	General Repairs and Maintenance.	Total Pumping Expense, including Salaries and Repairs.		
Springfield, Mass.	Gravity.	210[1]	1 460	1889	5.61	1.27	.... 0.92	....	7.80	15.7[3]
Fitchburg, Mass.	"	186[1]	480	1889	5.82	....	1.41	....	7.23	11.6[3]
Gloversville, N. Y.	"	250[1]	183[1]	1880	2.67	....	0.32	....	2.99	....
St. John's, N. B.	"	92	1 692	1888	3.39	....	0.49	....	3.88	....
Northampton, Mass	"	200[1]	55[1]	1889	15.45	3.66	.... 1.54	....	20.35	6.0[4]
Worcester, Mass.	"	170[1]	1 735	1889	1.98	....	3.17	....	5.15	17.6[3]
Hartford, Conn.	"	140	500[1]	1890	9.68	2.14	.... 4.06	....	15.88	10.0[7]
Little Falls, N. Y.	"	325	1 460[8]	1889	0.65	....	1.07	....	1.72	3.0
Johnstown, N. Y.	"	230	183[1]	1889	2.00	0.44	.... 0.81	....	3.25	15.0[4]
Bradford, Pa	"	219	409[1]	1889	0.88	0.39	.... 1.68	....	2.95	....
				Average	4.813	....	2.337	....	7.12	11.3
Fall River, Mass.	Pumping	185	685	1890	14.10	0.92	.... 1.58	1.25	17.85	29.1[3]
Cambridge, Mass.	to Reser-	65[1]	1 526	1889	5.56	0.28	.... 1.74	0.75	8.33	15.0[3]
Plymouth, Mass.[5]	voir.	84[1]	95	1889	4.42	1.70	.... 3.69	1.26	11.07	....
New Bedford, Mass.	"	128	1 298	1889	3.75	0.51	.... 0.60	0.80	5.66	....
Natick, Mass.	"	140[1]	93	1890	4.90	2.23	.... 0.30	1.93	9.36	....
Manchester, N. H.	"	144[1]	676	1889	5.32	....	1.86	0.67	7.85	26[3,7]
Troy, N. Y.	"	242	2 856	1890	0.56	....	0.70	1.21	2.47	5.0[4]
Hudson, N. Y.	"	..	553	1889	1.08[1]	0.36	.... 0.37	1.12	2.93	None.
Bridgeton, N. J.	"	60[1]	102	1889	4.35	1.04	.... 0.60	1.17	7.16	....
Nashville, Tenn.	"	308[1]	3 796	1889	1.06[1]	....	0.58	0.68	2.32	10.0
Newport, Ky.	"	260	475	1889	12.83	0.78	.... 1.26	2.72	17.59	15.0[7]
Malden, Mass.	"	130	365[1]	1889	7.50	1.09	.... 3.75	1.10	13.44	....
Lawrence, Mass.	"	161[1]	961	1889	8.11[9]	....	0.95	0.91	9.97	14.7[3]
Waltham, Mass.	"	192	208	1889	9.46	....	4.05	1.95	15 46	26. [7]
				Average	5.93	....	2.21	1.25	9.39	17.7
Taunton, Mass.	Direct	152	281	1889	9.4	1.20	.... 1.99	1.21	13.80	16.9[3]
Schenectady, N. Y.	Pumping	165	876	1889	0 47	....	0.63	1.60	2.70	....
Detroit, Mich.	"	49	12 875	1889	0.58	0.21	.... 0.12	0.48	1.39	10. [7]
Columbus, O.	"	187	2 259	1889	1.64	0.44	.... 0.26	0.70	3.04	6.0[4]
Atlanta, Ga.	"	322	757	1889	4.06[1]	....	1.34	2.33	7.73	17.0[6]
East Saginaw, Mich.	"	93[1]	1 151	1889	1.88	....	0.48	0.92	3.28	11.2[3]
Kalamazoo, Mich.	"	100	628	1888	0.0[2]	....	0.25	1.39	1.64	....
Mansfield, O.	"	230	457	1889	0.31	0.32	.... 0.99	0.97	2.59	....
				Average	2.29	....	1.03	1.20	4.52	12.2

1. Taken from other than official reports.
2. No debt.
3. Average of all water used through meters.
4. Minimum rate.
5. Part of supply by gravity (not given).
6. To all classes except manufacturers.
7. Doubtful.
8. Excess over consumption runs to waste.
9. Interest on first cost.

TABLE No. 3.

Number of Works Built in the United States and Percentage of Kind in Each Year.

Date.	Number of Each Kind Built.					Total Number.	Percentage of Each.					
	Gravity.	Direct Pumping.	Pumping to Reservoir.	Pumping to Stand-Pipe.	Pumping to Tank.		Gravity.	Direct Pumping.	Pumping to Reservoir.	Pumping to Stand-Pipe.	Pumping to Tank.	Pumping to Reservoir, Stand-Pipe and Tank.
Before 1840	15	1	7	0	2	25	.60	.04	.28	0	.08	.36
From 1840–49	4	0	10	0	0	14	.28½	.00	.71½	0	0	.71½
" 1850–59	22	3	15	1	1	42	.52	.07	.36	.02½	.02½	.41
" 1860–64	17	1	2	1	0	21	.81	.05	.09	.05	0	.14
" 1865–69	36	10	18	1	2	67	.53	.15	.27	.01½	.03½	.32
" 1870–74	69	42	43	8	10	172	.41	.24	.25	.04½	.05½	.35
" 1875–79	63	33	54	17	26	193	.33	.17	.28	.09	.13	.50
During 1880	17	8	7	3	0	35	.48½	.23	.20	.08½	0	.28½
" 1881	16	7	11	4	5	43	.37	.16	.25	.10	.12	.47
" 1882	24	23	13	15	11	86	.28	.27	.15	.17	.13	.45
" 1883	27	14	13	14	16	84	.32	.17	.15	.17	.19	.51
" 1884	25	19	30	19	19	112	.22	.17	.27	.17	.17	.61
" 1885	25	21	22	15	18	101	.25	.21	.21	.15	.18	.54
" 1886	24	20	38	36	21	139	.17	.14	.27	.26	.16	.69
" 1887	35	22	27	60	25	169	.21	.13	.16	.36	.14	.66
" 1888	38	26	40	81	53	238	.16	.11	.17	.34	.22	.73

pipe system becomes its natural development under such conditions, and we therefore find the percentage of the latter showing an inverse ratio with the direct system. That the stand-pipe modification is a favorite in small towns is seen by its frequent use in the later works, the greater part of the latter being of moderate size.

Disregarding the gravity systems, as being entirely dependent upon local conditions, the arguments in favor of the reservoir, as opposed to direct pumping, are usually based on the following suppositions:

First.—Better fire protection due to the storage of a large quantity of water.

Second.—Greater economy in running expenses with longer life to pump and boiler.

Third.—Purer water due to sedimentation in the reservoir.

With respect to superior fire protection the somewhat meager statistics in possession of the writer prove little, and that little is in favor of the direct system. Large towns that depend upon their pumps have always a duplicate plant that can be brought into service at a moment's notice, with a hydrant pressure that is usually limited only by the strength of the pipe system. The pump in daily use can be, and often is, run at its full and most economical capacity, giving a "life duty" that well compensates for its idle brother. Since the boilers supplying steam to the daily service pump are, as a rule, taxed only about 50 per cent. of their full capacity, the reserve pump can be brought into service at once. It can also be noted that the mere knowledge of the fact that the town is dependent upon the perfect working of the pump, and that even a stoppage of a few minutes will be commented upon and probably investigated, tends to a greater watchfulness and care on the part of the men in charge, with a saving of the proverbial nine stitches. The importance of the service also demands, and usually obtains, a better paid and more skillful class of labor.

As the consumption increases in amount the extra pump, being usually of greater capacity than that used in daily service, takes the place of the latter, and a new one is purchased to perform guardian duty. The old one then, if not worn out, can be sold for some fair percentage of its original cost, as it is still useful in many situations where a high duty is not required. It is thus seen that the purchase of a duplicate plant antedates but a few years the actual necessity, and is losing nothing but the interest of the investment.

When in a reservoir system the method is adopted of pumping but half the time, the outlay for engines will be at least equal to the duplicate plant of the direct system. The strains in the boilers induced by partial cooling each night, when the fires are banked, must have also an injurious effect upon their durability.

With regard to the purifying action of storage reservoirs the writer is of the opinion that, with the exception of the removal of the grosser particles, water is far more injured than improved by the process. The mud and slime that is deposited in the bottom undergoes decomposition whenever the weather permits, sustaining an infinite number of evil smelling plants. There is also usually a fouling of the whole body of water at each decided change of temperature.

The removal of the first cause, the mud, could more easily, and certainly more perfectly, be attained by some one of the many good forms of mechanical filters. A battery of these can be purchased for about one-quarter of the first cost of a reservoir, while they have the great advantage of permitting, with small waste of water, so frequent cleaning as to prevent the first beginning of decomposition.

If then, from the above data any conclusions can be drawn, they may be summarized as follows, supposing in every case that the three systems of supply are open to choice: Whatever the size of the town, a gravity system would be most efficient whenever these conditions are fulfilled—the supply of unquestioned present and future purity; the quantity sufficient for the needs of the next twenty years without great additional outlay; and the original cost such that 8 per cent. thereof will not be greater than 60 per cent. of the total operating expense, including interest and sinking fund.

A direct pumping system will be the most economical; when the town is of such a size that its daily consumption will be more than one-half million gallons; when the supply is good and abundant, but situated at a low level; and when there are no great differences of elevation in the distribution system. If there are highly elevated portions, these, when of small comparative area or consumption are best supplied with a reservoir system, having an independent force main. When the consumption is less than one-half million gallons, direct pumping with a reservoir-stand-pipe of at least twenty hours capacity, situated in the distribution system, will give the most economical results.

When a town is so unfortunate that its supply is insufficient during certain portions of the year, or when that supply is sufficient for ordinary needs, but affords no margin for fire protection, a reservoir of large capacity will then be necessary. Works of the latter kind can only be regarded as temporary in character, the inevitable growth of the city forcing soon radical changes and possibly reconstruction.

Although the writer is aware that no hard and fast rules can be formulated for a choice of a system, and that a hundred local circumstances will determine the final selection, yet he believes that a few general principles like the above will afford some aid to the solution of doubtful cases.

Before leaving the subject it may be well to call attention to the lack of accurate data as to the life of various parts of a water-works system. Although such data undoubtedly exist in the offices of each plant, they are only available to the profession through the brief and detached hints found in annual reports. Who can tell, except in individual cases, what is the average life of a pump, and how much has been spent on it for repairs, or under what condition of soil a cast-iron pipe will last five years or thirty?

The financial data furnished in annual reports are also capable of improvement. Not more than one-quarter of those in the possession of the writer were available in preparing Table No. 2, as many that gave full particulars of operating expenses neglected to state the consumption; no distinction was often made between renewals and extensions, and in a great number of cases no interest account was given. It is manifestly unfair, also, when comparing the efficiency of different towns, to charge a percentage sinking fund against an original cost, the major portion of which has long been paid.

DISCUSSION.

Desmond FitzGerald, M. Am. Soc. C. E.—I have been asked to open this discussion. It is a task I do not particularly enjoy, and should much prefer to have some one else begin, as I do not find myself at all in sympathy with this paper, and my remarks must necessarily be in the nature of criticism. I am very sorry that the author of the paper is not present, for it always seems to me that one can speak more unreservedly if the author is present to defend his osition; however, there

are certain duties that we owe to our profession. I must confess to disappointment in the data and results contained in Mr. FitzGerald's paper.

In Table No. 1 the Boston works, for instance, are classified under the head of "Pumping to Reservoir."

In round numbers, of the 40 000 000 of gallons supplied to Boston daily, there are about 14 000 000, including the Mystic works, so called, which are pumped. Our main supply is carried to the city by the Cochituate and Sudbury River aqueducts by gravitation, and then about 6 000 000 of gallons daily are pumped, within the city limits, to a high service reservoir with a separate distribution system, so that I have always been accustomed to look upon the Boston supply as principally a gravity system. Sixty-five per cent. of the consumption is gravity, pure and simple, and of the remaining 35 per cent. nearly one-half is brought from the sources by gravity before passing through the pumps. These facts are all set forth fully in our annual reports. Aside from errors such as this, I think there are some points made in the text of the paper, against which we should raise a mild protest. Is it possible, from facts like those contained in Table No. 1, to produce a set of hard and fast rules applicable to all or even a majority, of cases? It seems to me that we must consider each case of water supply under its own particular conditions, and that serious errors are liable to arise from lumped classification, taken from statistical tables. I should at least advise any who are just setting out in the water supply branch of the profession, to study very carefully the data contained in this paper before accepting the conclusions.

I notice on page 5 the life of pipe systems is set down as "certainly less than twenty years." I should think this a very short period to take as the life of a cast-iron pipe.

As to the effects of storage upon a water supply system, it strikes me that the writer has fallen into serious error. It will be found, I think, that the only waters which deteriorate on storage in properly prepared reservoirs, are those which are filtered or taken from subterranean sources. As is already well understood, these waters should be stored only in closed reservoirs. The storage in our Boston works amounts to about 6 000 000 000 of gallons, and in the management of this storage, I have had an opportunity to watch for a number of years the effects of storage. The results to which I have arrived are, that surface waters are never injured by proper storage; on the contrary, that the chances are in the great majority of cases that they will be very materially improved, and further, that the poorer the quality of the water, or its equivalent, the greater the amount of organic matter in process of change, the greater will be the benefits of storage.

I have abundant facts and figures to fortify this position, but this is hardly the proper opportunity to enter into an extended argument. I should like to state, however, that I have found most certainly,

that the different forms of nitrogen which represent the first steps in putrefaction, such as free and albuminoid ammonia and the nitrites, are lessened in proportion to the length of the time of storage; the color is lessened, and the impurities, so called, rendered more inocuous.

From long experience and many experiments on a large scale, I have found that the effects of storage are very different when a reservoir is properly prepared by removing the shallow flowage, and taking out the loam, muck, stumps and other objectionable matter from the bottom and sides of a valley, before flowing it with water for a domestic supply. If no pains are taken in the preparation of the basin or reservoir, if we store water over a swamp or other source of decaying vegetable matter, we must expect to pay the penalty. But I am afraid I am making my discussion too long and will close with the remark that there is one clause in the paper with which I can agree most heartily, and that is that "no hard and fast rules can be formulated for the choice of a system, and that a hundred local circumstances will determine the final selection."

James B. Francis, Past President Am. Soc. C. E.—I should like to know under what condition of soil the cast-iron pipe will last only five years. It is within my experience and knowledge that pipes at Lowell, which were obtained from Philadelphia in 1830, were then intended to be used under a pressure of 150 feet head, and are now in use under 200 feet head, without any signs of failure after sixty years' use.

F. L. Fuller, M. Am. Soc. C. E.—I question the statement that the life of pipe systems is not more than twenty years. I think there are a great many pipe systems that have stood longer than that; and also the statement that many water companies sell water below the cost of production.

Charles B. Brush, Director Am. Soc. C. E.—I agree very heartily with the remarks made by Mr. Desmond FitzGerald in relation to this paper, and I agree with one paragraph of the paper, viz.: "It may be well to call attention to the lack of data as to the life of systems, although such data undoubtedly exist." The records in this particular are certainly very brief and unsatisfactory, as they are usually in relation to other important details affecting the operation of waterworks; and it is from just such unsatisfactory records that these data have been collected. Mr. FitzGerald has referred to the different conditions under which a supply is obtained at different points. On one of the works under my control the force main is 17 miles long, on another it is 22 miles long, on another it is only 3 miles long. On one of these works it is necessary to pump the water three times before it reaches the consumer; on the other works it is pumped direct to the consumer. There is no relation whatever between systems which have only a low service and those which have a low and high service. The data upon which this paper has been prepared, I think are utterly inadequate as

a basis for any safe conclusions. The conclusion as drawn seems to me rather peculiar. It is true that in some instances a direct pumping system is perhaps desirable, yet it is strange how many towns, after they have established a direct pumping system, have found it necessary to change, and those changes are being made all over the country. The inference naturally drawn from the paper before us would seem to be, that a direct pumping system was much the most economical. That has not been my experience, and I think before any such general principle is laid down, very much more careful and full investigation should be made. I object to any such conclusion going forth from this Society.

The extreme draft on different systems varies as compared with the average draft, but it is safe to say that the draft in the middle of the day, from 9 to 3 o'clock, is often three times the average draft of any given plant. If you have a direct system you must have at least a pumping capacity equal to the extreme draft; and the pumping capacity will have to be at least three times as great as the pumping capacity on the reservoir system. That carries with it not only the pumps but also the boilers and appurtenances. The strain on a plant which is supplied by direct pressure is always a very serious matter. I have compared the hourly draft of different systems, and I have found the loss greater on a direct system than on a reservoir system.

My own experience is directly the reverse of the conclusions of the writer, and I certainly would not be willing to accept his conclusions unless the data were very much more satisfactory than the data submitted in this paper.

J. B. JOHNSON, M. Am. Soc. C. E.—I would like to enter my protest against any kind of an argument based on averages where the average is of this kind; this average is the straight arithmetic mean of all the quantities in the table, giving equal weight to all. Thus, for instance, under Pumping to Reservoir, Table No. 1, I take City of Pittsburgh that supplies 30 000 000 gallons a day at a cost of 5.9 cents per 1 000 gallons; he gives that no more weight in his average than the City of Newport, Ky., which pumps 900 000 gallons a day at a cost of 31 cents per 1 000 gallons. In the first place the conditions and the items are so various that an average means nothing; an average would mean nothing even though the cities were all of the same size, but when one city uses fifty times as much water as another and has fifty times larger works, he has given them both exactly the same weight, simply taking an average, counting each city as one. An average like that is worse than nothing, it is misleading. Engineers should know better, for they know that one fact is more significant than another. As he acknowledges that his conclusions are drawn from these tables and not from what his judgment might lead him to conclude from the facts, I submit if this is not absolutely a false basis on which to base an argument for engineers to

use, in settling so important a problem as deciding what kind of a system to use for the water-works of a city.

T. H. McCann, M. Am. Soc. C. E.—Mr. FitzGerald says that there are probably misstatements in his tables; I find this to be the case as regards the City of Hoboken, N. J.

The true statements as to this city are as follows: From 1855 to 1883 Hoboken was a portion of the distribution system of Jersey City. Since 1883 water has been supplied by a private company (The Hackensack Water Company), the supply being paid for by house rates, which are collected by the city authorities and paid to the company, less a percentage to pay for expenses of care of mains, cost of collecting, etc.

The present population is about 43 000 instead of 75 000, and the present length of mains is 20 miles instead of 78.50 miles as given in the tables.

Joseph Ramsey, Jr., M. Am. S. C. E.—I come from a city where to-day I think we are getting posted on water-works, that is, Cincinnati. There we have water-works, but no water. Mr. FitzGerald is not here, and we seem to have discovered it somehow; I dislike to see a man pitched into when he is not present to defend his paper; Mr. FitzGerald's general deductions probably are not altogether free from the errors which follow too broad an application of averages, for there is a rule that "averages will lie sometimes," but I think he has given some tables which are of great value. The tables made from the reports of all the various companies are very interesting. Of course in some cities where they have combined systems, such as Boston and some other places, he may be mistaken in his conclusions, but any one willing to make proper allowance for the varying conditions governing the operation of the water-works of each of these cities, can take these tables and make his own deductions. I think this paper is worthy of some credit, and for the tables he has given us Mr. FitzGerald certainly deserves our thanks.

The Chair (William Metcalf, M. Am. Soc. C. E.).—If there is no further discussion I want to mention one matter that is to me of great interest. It always did seem to me that it was extremely dangerous for a town to depend upon a pump if it was possible to get a reservoir.

We have had at times a great deal of trouble from the floods in the Allegheny River, and a couple of years ago I thought I would make a great improvement. I had the Philadelphia Company send their dredge up and dredge a place some 10 or 15 feet deep in the bed, and near the mouth of the Allegheny River. I then had a heavy timber crib built in the space thus dredged, and sunk our suction pipes into this crib. We got a beautifully clear water, and thought we had done a very great thing, but in a few days our whole concern was up in arms. A great many complaints were made, and they told me they could not get the boilers clean. The man in charge of the boilers said if he had to use

that water he would give up the job because he knew the result would be an explosion. Of course I thought they were simply pumping out a little loose sand, but I had Professor Langley take up the matter and analyze the water to see if there was any real cause for trouble. The Allegheny River water is a very soft, delicious water, and we found that we had in that short distance of 12 or 15 feet struck a sub-river of lime-water some 12 feet below the Allegheny, which contained thirteen times as much sediment as the muddiest river-water we could get from a dirty flood stage in the river. So we were obliged to destroy the crib at considerable expense.

J. LELAND FITZGERALD, M. Am. Soc. C. E.—It was my intention in the foregoing paper to confine the arguments to the scope of the title, that is, which class of systems is the cheapest based on the data at hand. If no reliance can be placed on the latter, no general conclusions can be drawn.

As to the average life of cast-iron mains, I confess that I have been able to procure no data on the subject. In Schenectady the mains were originally of cement-lined wrought-iron and were laid in 1870. They have almost entirely been replaced by cast-iron, and all extensions since that date have been of the latter material. Many of these have been relaid but, as much on account of insufficient size as of growing weakness of material.

From the conflicting facts as to the effects of storage on water, the subject can hardly be called settled. Mr. L. J. LeConte, in a paper read before the American Water Works Association, discusses the effect of storage of surface waters on a large scale on the Pacific Coast. In addition to this, nearly every report from towns having a reservoir system, speaks of there being trouble at some time or other from low growths. Rain water is stored in some Mexican towns in large quantities and in covered reservoirs. The water at the end of the dry season grows so foul that strangers drink it with the greatest repugnance.

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DETERMINATION OF THE STRESSES IN ELASTIC SYSTEMS BY THE METHOD OF LEAST WORK.

By WILLIAM CAIN, M. Am. Soc. C. E.

A new method of ascertaining the stresses in elastic structures, by means of the principle of Least Work, has been elaborated by Castigliano, in his "Théorie de l'Equilibre des Systémes Élastiques" (Turin, 1879), who claims to have given the first complete demonstration of the theorem of least work, though several authors have touched upon the subject from 1818 to the present time.

The method is found to be of very general application to all structures in which the laws of elasticity have to be considered in finding the stresses; as in every kind of beam or arch, trusses of any shape with superfluous members and all systems where there is any continuity in the members or where there is not free play at the joints, as in nearly all roof or bridge trusses.

It further offers an exact method by which we can ascertain the limit of error made in our ordinary approximate computations (which apply only to articulated systems, free to move at all the joints), and thus exposes some of the unknown errors which are usually included in our

"factor of safety," though it has more appropriately been termed our "factor of ignorance."

In the first nine articles that follow, the theorems of "least work" and "deflection" are deduced by aid of the principle of "virtual velocities as applied to framed structures," as developed by Mohr, Maxwell and others. The rather obvious relation between the two has, doubtless, been hitherto observed, though the writer has not met with it anywhere.

The demonstrations of Castigliano are very direct and elegant; but the method of virtual velocities, which is valuable in itself, leads so easily and unmistakably to his conclusions that it was preferred to his demonstrations, particularly as it leaves no doubt as to the exact interpretation of results. A *résumé* of the method of virtual velocities as applied to certain framed structures was given by Prof. George F. Swain in the *Journal of the Franklin Institute* for February, March and April, 1883, to which the writer is pleased to acknowledge his indebtedness.

In Articles 10, 11, 12, that follow, are given certain essential theorems (taken mainly from Castigliano) necessary to put the reader, in the briefest time, in possession of the general method leading to the applications. The theory of arches is then carefully developed in Article 13, and pains have been taken to make it as clear and simple as possible.

As Castigliano's book is accessible to engineers, it was thought best to refer to it for the numerical illustrations of the theory, which are given there in great number and variety. Some of the methods and results are briefly stated in Article 14, however, referring to "Composite Structures," and a comparison instituted between these correct results and those given by approximate methods, which should command careful attention. In fact, such comparisons should be the main object in view in cases where the exact and approximate methods do not differ too widely, for although the method of least work leads to a direct and systematized method of procedure that is easily acquired, yet the numerical applications are long, and will hardly come into daily use for such cases. Where the exact and approximate methods differ widely, the method of work should be used.

Section 1.

As preliminary to the treatment of elastic systems, we shall first state the well-known relation between the stress and the change of length of a prismatic bar subjected to simple tension or compression in the

direction of its length, and then deduce an expression for its elastic work of deformation during the change of length.

If we call a the length of the bar, ω its cross-section, e the modulus of elasticity and λ its change of length under the stress s, then we know from experience that when the resultant stress acts along the axis and the limit of elasticity has not been exceeded, that,

$$\lambda = \frac{as}{e\omega} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If, for the sake of brevity, we put $c = \frac{a}{e\omega}$, we have the alteration in length given by the following expression:

$$\lambda = cs \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where several bars are considered at the same time, we shall designate the corresponding quantities that enter in (1) and (2) by proper subscripts.

Now, if the bar lengthens or shortens the amount λ by the action of a load P acting in the direction of its axis, which gradually increases from 0 to its greatest value P, the stress likewise increases gradually from 0 to its greatest value s, and at any instant the stress is exactly equal to the load. Then if successive values of s, beginning with 0, are so taken that each increment of stress corresponds to equal changes of length ($d\lambda$), we have the total work of deformation equal to the limit of the sum,

$$\sum sd\,\lambda = \sum \left(\frac{e\omega}{a}\, \lambda d\lambda \right)$$

between the extreme valves $\lambda = 0$ and $\lambda = \lambda$.

$\therefore$ The work of deformation is equal to,

$$\frac{e\omega}{a} \int_{o}^{\lambda} \lambda\, d\lambda = \frac{e\omega}{a}\, \frac{\lambda^2}{2} = \tfrac{1}{2}\, \frac{a}{e\omega}\, s^2 = \tfrac{1}{2}\, s\lambda \quad \ldots\ldots\ldots (3)$$

or since load $P = s$, work $= \frac{1}{2}$ load $\times$ change of length. For a suddenly applied load P, causing a change of length λ^1 and maximum stress s, the work of the load is $P\lambda^1$, and since the stress gradually increases from 0 to s, the work of deformation by (3) is $\frac{1}{2}s\lambda^1$, and since these are equal, $s = 2P$, or the maximum stress, is double the load, hence by (1) the change of length is double that for a gradually applied load. After a series of oscillations this change ultimately becomes that due to a gradually applied load, and s reduces to P.

Section 2.—The Law of Virtual Velocities.

If the point of action of a force is supposed displaced through an indefinitely small distance, the projection of this distance on the direction of the force is called its virtual velocity.

If the direction of this projection, counting from the original point of action of the force, is in the direction of the force, *i. e.*, if the force moves forward as it is directed, the virtual velocity will be called plus; if the direction of the projection of the displacement is opposed to that of the force or if the force is moved backward or contrary to the direction in which it tends to go, then the virtual velocity will be called minus. For the present all forces will be regarded as plus.

With the above conventions assumed, the principle of virtual velocities consists in this: that if any number of forces, at one or more points of a rigid body, be in equilibrium, then if the body is supposed to receive an indefinitely small displacement, the algebraic sum of the products of each force into its virtual velocity is equal to zero.

The product of a force by the distance through which it is moved, counted along its direction, is called the work of the force, which is positive or negative according as the virtual velocity is positive or negative.

Section 3.

As trusses with superfluous bars will be considered in what follows, it is well, likewise, to recall, that where the figure of a truss has more lines than are necessary to strictly define its form, *i. e.*, to fix its apices, when the length of sides are given in order, the extra sides are said to be "superfluous."

The relation between the least number of sides m, or the number of "necessary" bars in a truss, and the number of joints or apices n, for strictly defining the form of a figure of invariable form, is easily arrived at.

Thus, for plane figures, assume the position of one side, thus fixing the two apices at its ends. From these apices we can fix another with two new sides, then another with two new sides from two apices previously fixed, and so on; therefore, to each of the $(n - 2)$ joints other than the first two, corresponds two sides, so that the total number of necessary sides $m = 2\,(n - 2) + 1 = 2n - 3$. If the number of sides exceeds $(2\,n - 3)$, the extra number are "superfluous" to strictly define the

form. A less number will give a figure that can change its shape without changing the lengths of its sides.

For figures in space, we select three sides in position to fix three apices, then to fix each new apex, we need three sides or distances from three apices already fixed. Therefore it takes three sides for each apex, except the first three, which required only three in all. Therefore, $m = 3(n - 3) + 3 = 3n - 6$, which gives the least number of sides to constitute a figure of invariable form in space.

Now, in any truss, without superfluous bars, constituting a figure of invariable form, statics can determine the stresses in all the members; for at each of the n joints, for trusses in one plane, we can write two equations, expressing the law that the sum of the horizontal as well as of the vertical components of the external forces and stresses in the bars, meeting there, are separately equal to zero. This gives $2n$ equations, but these reduce to $(2n - 3)$ independent equations for trusses in one plane, or the exact number of bars, since there are three necessary equations between the equilibrated external forces, indicating that the sum of their horizontal components is zero, the sum of their vertical components is zero, and the sum of the moments of the forces about any point in the plane of the forces, is zero.

For trusses having any position in space, statics gives at each joint three equations, expressing that the sum of the components of external forces and stresses there, parallel to three axes, are separately equal to zero, giving in all $3n$ equations; but we have now 6 equations of equilibrium between the external forces alone, hence the number of independent equations between the stresses reduces to $(3n - 6)$ or the number of the bars.

In both cases, then, of trusses in one plane or having any position in space, we have only as many equations as bars, when there are no superfluous members, so that the stresses in the bars can be found. If the truss, therefore, has superfluous bars, statics will furnish too few equations by the number of superfluous bars.

Section 4.—Application of the Principle of Virtual Velocities.

Let us consider the frame, Fig. 1, lying in one plane, free to move at the joints, with smooth supports and having one superfluous bar, say CD, since $m = 6$ exceeds $2n - 3 = 8 - 3 = 5$, by one.

Here statics alone cannot determine the stresses, giving too few equations by one:

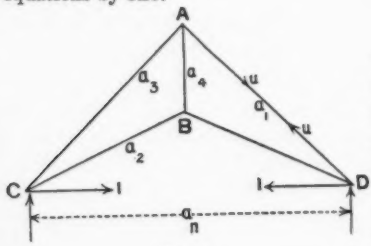

Fig. 1

Let us first suppose no load on the truss, that bar CD is removed, and that two opposed forces each equal to unity, act toward each other at C and D along the line CD.

These forces will generally cause stresses in the remaining bars, and reactions at the supports, unless they are smooth as assumed; that can be computed by statics alone. Call the stress induced by the forces 1 acting along CD in bar AD, u and the change of length of AD, Δl.

If the forces unity cause tension in AD, we can suppose the bar AD removed if we replace its action by two forces each equal to u acting toward each other at A and D. The system with the bars CD and AD removed and the forces 1 and u applied as shown, constitutes a system in equilibrium.

Now from whatever cause, suppose the distance CD to be increased an amount Δp^1 due to an increase Δl in AD alone, the remaining bars retaining their original lengths. The virtual velocities of forces 1 and u are both negative as by Section 2, the forces are moved contrary to their directions. Therefore by the principle of virtual velocities,

$$-1\,\Delta p^1 - u\Delta l = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The value of the elongation Δp^1 from this equation is $(-u\Delta l)$, so that the points C and D must approach each other when A and D recede from each other, as would happen when u is tension. It is convenient, however, throughout, when we do not know the character of the stresses beforehand, to assume the stresses in all the bars to be tensions, and the corresponding change of length between any two apices to be an increase. On solving the resulting equations to be presently deduced, the true character of the stresses will depend upon the sign of the resulting stress, plus for tension (as assumed), minus for compression.

If the two forces unity, acting along CD, cause compression in the bar AD, before it it supposed removed, then the two forces u at A and D that replace the action of the bar AD upon the rest of the truss,

will act in the opposite directions to those shown in the figure, and the term $(-u\Delta l)$ in (4) becomes $(+u\Delta l)$ and $\Delta p^1 = +u\Delta l$ is an elongation as assumed.

We see, therefore, that u, in equation (4), is to be taken as $+$ or $-$ according as it represents tension or compression.

Now, as there has been no restriction whatever as to the numerical value of Δl, it can be taken equal to the actual change of length of bar AD due to the loading assumed for the complete truss bars AD and CD included. Hence, if we call a_1 the length of bar AD, e_1 its modulus, ω_1 its cross-section and s_1 the actual stress in it, due to the actual loading on the complete truss, we have:

$$\Delta l = \frac{a_1 s_1}{e_1 \omega_1} = c_1 s_1$$

So that equation (4) becomes

$$-1.\ \Delta p^1 - u\frac{a_1 s_1}{e_1 \omega_1} = 0,$$

in which if s_1 is tension, it is to be given the $+$ sign, if compression the $-$ sign; since in the first case Δl is an elongation as assumed, and in the second case a shortening, so that the principle of virtual velocities, as illustrated by equation (4), obtains with the conventions adopted as to signs, the second term of the first member changing sign with s_1. The equation is therefore perfectly general, and it is only necessary to pay proper attention to the signs of the stresses and displacements, to correctly interpret the results.

We can now write an equation similar to the last one above for the remaining bars of the truss, by considering the change in length in CD due to each bar in turn, supposed elastic, the remaining bars being temporarily supposed inelastic, since in the truss without the one superfluous bar CD, any bar can change length without causing any change of length in the other bars.

Therefore regarding all the bars as elastic we have the total change of length of CD (plus for an elongation, minus for a shortening) due to the changes of length of all the bars other than CD, equal to

$$\Delta p = -\sum u\frac{as}{e\omega} \qquad (5)$$

But this change of length Δp in the distance CD, due to the changes in length of the other bars, resulting from the assumed loading of the complete truss, is exactly equal to the real change of length in the super-

fluous bar *CD*, due to the same loading, since the final lengths and positions of the other bars, completely fix the new length *CD* of the superfluous bar after strain, in consequence of the necessary geometrical relation between the lengths (Section 3).

Therefore, if s_n is the stress caused in the superfluous bar by the loading, and a_n, e_n and ω_n represent the length, modulus and cross-section respectively, we have,

$$\Delta p = \frac{a_n s_n}{e_n \omega_n},$$

whence,

$$\frac{a_n s_n}{e_n \omega_n} + \sum u \frac{as}{e\omega} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

the sum being extended to all the bars other than *CD* or the "necessary bars" as they will be called, as this number of bars is necessary to define the form of the truss, any other bars being superfluous to that end.

If we designate generally by y, the stress caused in any one of the necessary bars by the loading assumed, when the superfluous bar is removed, which stress can be computed by statics alone (as also can u, the stress caused by forces 1 acting toward each other in the line of the removed superfluous bar, whose real stress is s_n), then by the principle of the superposition of effects, the actual stress in the necessary bar considered is,

$$s = y + u\, s_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

On substituting such values for s in equation (6) for all the necessary bars, we have an equation in which the only unknown quantity is s_n (the stress in the superfluous bar), which can thus be found, and afterward the stresses in the remaining bars can be determined by aid of equation (7).

Strict attention must be paid to the signs of the stresses u (tension +, compression —) in equations (5) and (6) as already mentioned, whence the resulting sign of s_n and of the stresses in the necessary bars will indicate the character of the actual stresses in all the bars due to the assumed loading.

If we term the $n-1$ bars, to be numbered 1, 2, ... $(n-1)$ on the diagram, the necessary bars, then the n^{th} bar will be the superfluous one, and designating by the proper subscripts the constants a, e, ω and u, pertaining to each bar, equation (6) can be written as follows, for any truss whatsoever, having but one superfluous bar:

$$u_1 \frac{a_1 s_1}{e_1 \omega_1} + u_2 \frac{a_2 s}{e_2\ \omega_1} + \ldots + \frac{a_n s_n}{e_n \omega_n} = 0 \ldots\ldots\ldots\ldots (8)$$

where $s_1, s_2 \ldots$. have values given by expressions of the form shown in equation (7). It is well to observe that the demonstration given is equally correct, whether the truss is supposed in one plane or not.

Example.—A simple example will illustrate the working of the method.

In Fig. 2, the frame is supposed to consist of three bars) 1, 2 and 3), whose lengths are a_1, a_2, a_3, respectively, the bar AB being taken as the superfluous one. The distance $BC = CD = b$; the actual stresses due to load W at A are s_1, s_2, s_3, in bars 1, 2, 3, the moduli e_1, e_2, e_3, and the cross-sections ω_1, ω_2, ω_3, respectively.

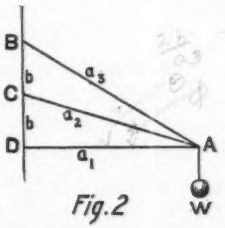

Fig. 2

Now suppose the bar AB removed, and that forces equal to unity act towards each other at A and B, the weight W being removed. Calling the stresses induced in bars 1 and 2, by these forces unity, u_1, and u_2, we have (treating the stresses as tensions until ascertained), on balancing vertical and horizontal components separately at the apex A,

$$1\,\frac{2b}{a_3} + u_2\,\frac{b}{a_2} = 0$$

$$1\,\frac{a_1}{a_3} + u_2\,\frac{a_1}{a_2} + u_1 = 0$$

Whence—

$$u_2 = -\,\frac{2a_2}{a_3},\quad u_1 = \frac{a_1}{a_3}$$

Similarly, calling Y_1 and Y_2 the stresses caused in bars 1 and 2 by weight W, when bar AB is supposed removed, we find,

$$Y_1 = -\,W\frac{a_1}{b},\quad Y_2 = W\frac{a_2}{b}$$

Therefore, the actual stresses in bars 1 and 2 are—

$$s_1 = y_1 + u_1 s_3 = -\,W\,\frac{a_1}{b} + \frac{a_1}{a_3}\,s_3$$

$$s_2 = y_2 + u_2 s_3 = W\frac{a_2}{b} - \frac{2a_2}{a_3}\,s_3$$

We can at once substitute these values in equation (8) and solve for $s_3 = s_3$ and then find s_1 and s_2 from the eqnations above; we have therefore

$$u_1\,\frac{a_1 s_1}{e_1 \omega_1} + u_2\,\frac{a_2 s_2}{e_2 \omega_2} + \frac{a_3 s_3}{e_3 \omega_3} = 0,$$

from whence by substitution and reduction we obtain finally, putting $e_1 \omega_1 = e_2 \omega_2 = e_3 \omega_3$ for simplicity,

$$s_3 = W \frac{a_3}{b} \cdot \frac{a_1^3 + 2a_2^3}{a_1^3 + 4a_2^3 + a_3^3}.$$

This stress is always +, so that bar 3 is always in tension as assumed. The stresses in bars 1 and 2 can now be found as indicated, and their signs will designate their character.

Section 5.—The Principle of Least Work.

Let us consider a frame of n bars, free to move at the joints, having one superfluous bar. Number the bars consecutively from 1, so that the nth bar is the superfluous bar.

Designating, as in the preceding article, by the proper subscripts, the constants a, e, ω and u pertaining to each bar, and writing for brevity,

$$\frac{a_1}{e_1 \omega_1} = c_1, \quad \frac{a_2}{e \omega_2} = c_2 \ldots\ldots;$$

then equation (8) can be written,

$$u_1 c_1 s_1 + u_2 c_2 s_2 + u_3 c_3 s_3 + \ldots\ldots + c_n s_n = 0 \ldots\ldots\ldots\ldots (9)$$

We likewise have,

$$\left.\begin{aligned} s_1 &= Y_1 + u_1 s_n \quad \therefore \frac{ds_1}{ds_n} = u_1 \\ s_2 &= Y_2 + u_2 s_n \quad \therefore \frac{ds_2}{ds_n} = u_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots \\ s_{n-1} &= Y_{n-1} + u_{n-1} s_n \therefore \frac{ds_{n-1}}{ds_n} = u_{n-1} \end{aligned}\right\} \ldots\ldots\ldots (10)$$

where $\frac{ds_1}{ds_n}$, $\frac{ds_2}{ds_n}$ ...., are the derivatives with respect to s_n of s_1, s_2 ...., s_{n-1}.

As Y_1, Y_2 .... Y_{n-1} are not functions of s_n, they disappear in the differentiation.

Therefore (9) can be written,

$$c_1 s_1 \frac{ds_1}{ds_n} + c_2 s_2 \frac{ds_2}{ds_n} + \ldots\ldots + c_{n-1} s_{n-1} \frac{ds_{n-1}}{ds_n} + c_n s_n = 0 \ldots (11)$$

which is the differential co-efficient, with respect to s_n, put equal to zero, of the expression,

$$F = \tfrac{1}{2} [c_1 s_1^2 + c_2 s_2^2 + \ldots\ldots + c_{n-1} s^2_{n-1} + c_n s^2_n],$$

or its equivalent,

$$F = \tfrac{1}{2} \left[\frac{a_1}{e_1 \omega_1} s_1^2 + \frac{a_2}{e_2 \omega_{2-1}} s_2^2 + \ldots\ldots + \frac{a_{n-1}}{e_{n-1} \omega_{n-1}} s^2_{n-1} + \frac{a_n}{e_n \omega_n} s^2_n \right] (12)$$

which by Section 1 represents the total work of deformation of all the bars that constitute the complete truss under the assumed loading. We therefore conclude, for a frame with one superfluous bar, that *the stresses of the bars after deformation are such as render a minimum the work of deformation of the system, having regard to the laws of statics.*

In order to make use of equation (11) to determine the stress s_n, from which $s_1, s_2 \ldots$ can be found from (10), we must express the other stresses $s_1, s_2 \ldots s_{n-1}$ as functions of s_n, so as to effect the differentiation involved. When the values of $s_1, s_2 \ldots$ are given, as in (10), this is quickly done, as $Y_1, Y_2 \ldots$, not being functions of s_n, disappear in the differentiation; but if the values of $s_1, s_2 \ldots$ are first found as functions both of s_n and some of the other stresses, these last are functions of s_n, and must be so expressed before the differentiation can be effected.

It will be observed that equation (11) is identical with equation (8) of Section 4, so that the solution is the same for any special case.

We shall now give two examples to show the application of the principle of least work.

Example I.—The truss is shown in Fig. 1, Section 4. Call $2a_1$ the length of span $= 2\,000$, $a_2 = 1\,118$, $a_3 = 1\,414$, $a_4 = 500 =$ half the height of truss, and suppose the truss loaded at the upper apex with 2 tons, giving vertical reactions of 1 ton each, with a supposed perfect roller under one end. The stress under this load in bar CD will be called s_1, in bars CB or BD s_2, in bars AC or AD s_3, and in bar AB s_4, the bars symmetrical with respect to the center having the same stress.

Then regarding all stresses as tensions, we can deduce the following equations by noting that the sum of the horizontal and vertical components of the forces acting at C are separately equal to zero, and by balancing the vertical components of the stresses in the bars meeting at B.

$$1 + \frac{500}{1\,118} s_2 + \frac{1\,000}{1\,414} s_3 = 0 \ldots\ldots (1)$$

$$s_1 + \frac{1\,000}{1\,118} s_2 + \frac{1\,000}{1\,414} s_3 = 0 \ldots\ldots (2)$$

$$2 \frac{500}{1\,118} s_2 - s_4 = 0 \ldots\ldots (3)$$

From these three equations we derive:

$$s_1 = \left(1 - \frac{500}{1\,118} s_2\right), \qquad \therefore \frac{ds_1}{ds_2} = -\frac{500}{1\,118}$$

$$s_3 = -\left(\frac{1\,414}{1\,000} + \frac{1\,414}{2\,236}\, s_2\right), \quad \therefore \frac{ds_3}{ds_2} = -\frac{1\,414}{2\,236}$$

$$s_4 = \frac{1\,000}{1\,118}\, s_2, \qquad \therefore \frac{ds_4}{ds_2} = \frac{1\,000}{1\,118}$$

We have thus expressed s_1, s_3, s_4, in terms of s_2, though any other bar than BC could have been selected as the superfluous bar with equal propriety, the ease of elimination being the chief thing to consider. Now, if for simplicity, we regard all the bars as having the same modulus of elasticity and cross-section, we have the total elastic work of deformation after strain, for all the bars of the truss,

$$L = \frac{1}{2e\omega}\left[2a_1\, s_1^2 + 2a_2\, s_2^2 + 2a_3\, s_3^2 + a_4\, s_4^2\right],$$

which must be a minimum; therefore putting its first differential coefficient, with respect to s_2, equal to zero, we have,

$$2a_1\, s_1 \frac{ds_1}{ds_2} + 2a_2\, s_2 + 2a_3\, s_3 \frac{ds_3}{ds_2} + a_4 \frac{ds_4}{ds_2} = 0$$

On substituting the above values for s_1, s_3, s_4, $\frac{ds_1}{ds_2}$, $\frac{ds_3}{ds_2}$, $\frac{ds_4}{ds_2}$, we readily deduce from the resulting equation of the first degree as to s_2,

$$s_2 = -.39 \text{ tons, or } .39 \text{ tons compression,}$$

whence, substituting in the above values for s_1, s_3, s_4, we find,

$$s_1 = +1.17,\ s_3 = -1.17,\ s_4 = -0.35.$$

For span invariable (no roller) the term $2a_1\, s_1^2$ drops out of the value of L as the horizontal bar is no longer in action, and the abutments supply horizontal forces acting inward, which may be termed s_1, so that equations (1), (2) and (3) still hold on this supposition. As the work of the bars in action must be a minimum, as before $\frac{dL}{ds_2} = O$, which gives us,

$$s_2 = -.67 \text{ ton} = 0.67 \text{ ton compression.}$$

On substituting this value, in the expressions for s_1, s_3, s_4, we find,

$s_1 = +1.306$ tons = horizontal component of reaction,

1 ton = vertical component of reaction,

$s_3 = -1.00$ ton (compression),

$s_4 = -0.60$ ton (compression).

Compare with the solution given on page 330 of *Van Nostrand's Magazine* for October, 1882, by another method.

Example II.—In the truss shown in Fig. 3, if we suppose the diagonals to be equally capable of taking tension or compression, the truss

has one superfluous bar, as $m = 10$ is greater by 1 than $2n - 3 = 2 \times 6 - 3 = 9$. Let the lengths and stresses be designated as in the figure, and for simplicity, suppose a common modulus and cross-section for the bars included in the investigation below. If the upper segments of the chord are taken equal, the right reaction $= Q = \frac{1}{3} P$, the left reaction $= \frac{2}{3} P$. From these we can at once derive the stresses in the outer triangles so that they can be treated as external forces and we can confine our investigation to finding the stresses in the six remaining bars of the central portion A, B, C, D.

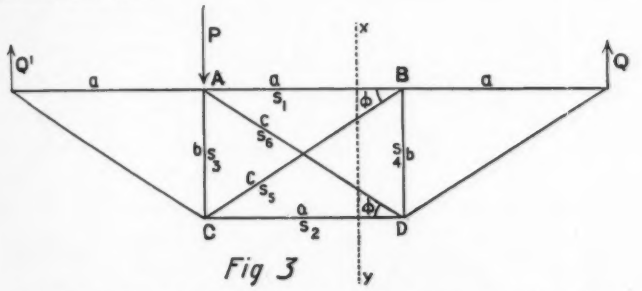

Fig 3

Statics can furnish five independent equations, giving the relations between the stresses in the six bars, and the principle of least work will furnish the remaining equation needed.

Thus, suppose a section $x\ y$, cutting the four bars shown, and that the part of the truss to the right of the section is in equilibrium under the action of the stresses in the four bars and of the right reaction Q. Then, calling the angle $ABC = \varphi$, and taking moments about B, regarding all stresses as tensions, we have,

$$(s_2 + s_6 \cos.\ \phi)\ b - Qa = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly taking moments about D,

$$(s_1 + s_5 \cos.\ \phi)\ b + Qa = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Next, balance the vertical components of the stresses at the section $x\ y$, with the reaction Q,

$$(s_5 - s_6) \sin.\ \phi - Q = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Also, since the sum of the vertical components of the stresses meeting at the point B is zero,

$$s_5 \sin.\ \phi + s_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Similarly for the point A,

$$s_6 \sin.\ \phi + s_3 + P = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Now, treating AD, as the superfluous bar, we must express the other stresses in terms of s_6, which is easily done by elimination in the preceding equations, and then the derivatives given are easily found.

$$s_1 = -\left(\frac{Qa}{b} + Q \cot.\ \phi + s_6 \cos.\ \phi\right) \quad \therefore \frac{ds_1}{ds_6} = -\cos.\ \phi$$

$$s_2 = \frac{Qa}{b} - s_6 \cos.\ \phi \quad \therefore \frac{ds_2}{ds_6} = -\cos.\ \phi$$

$$s_3 = -(P + s_6 \sin.\ \phi) \quad \therefore \frac{ds_3}{ds_6} = -\sin.\ \phi$$

$$s_4 = -(Q + s_6 \sin.\ \phi) \quad \therefore \frac{ds_4}{ds_6} = -\sin.\ \phi$$

$$s_5 = \frac{Q}{\sin.\ \phi} + s_6 \quad \therefore \frac{ds_5}{ds_6} = 1$$

The total elastic work due to deformation of the six bars is,

$$L = \frac{1}{2e\omega}\left[(s_1^2 + s_2^2)\, a + (s_3^2 + s_4^2)\, b + (s_5^2 + s_6^2)\, c\right]$$

Placing the derivative, with respect to s_6, equal to zero, we deduce,

$$\left(s_1 \frac{ds_1}{ds_6} + s_2 \frac{ds_2}{ds_6}\right) a + \left(s_3 \frac{ds_3}{ds_6} + s_4 \frac{ds_4}{ds_6}\right) b + \left(s_5 \frac{ds_5}{ds_6} + s_6\right) c = 0$$

On substituting the above values, and solving for s_6, we derive, after putting $\sin.\ \phi = \frac{b}{c}$, $\cos.\ \phi = \frac{a}{c}$, $\cot.\ \phi = \frac{a}{b}$, $\operatorname{cosec.}\ \phi = \frac{c}{b}$ and $Q = \frac{1}{2} P$,

$$s_6 = -\frac{Pc}{6b} \cdot \frac{a^3 + 4b^3 + c^3}{a^3 + b^3 + c^3};$$

from whence, by aid of the previous equations, we can find all the other unknown stresses.

In case the sections and moduli are not taken equal for the six bars, the solution is equally direct, but the value of s_6 is given by a much more complicated expression.

As is well known, American engineers generally construct the diagonals so that they can take but one kind of stress; therefore, as there is no superfluous bar under strain, statics alone can determine the stresses and the previous method is not needed.

As already mentioned, the solutions just given are identical with the method of section 4, so that so far there is no saving in labor in using the method of least work; but the rule is so simple and the principle so interesting from a scientific point of view, that possibly many will prefer it to the method on which we have founded it.

It is easily seen from these examples that the values of u of Section 4 may be obtained, as can the values of $\frac{ds}{ds_n}$ from the values of s above, by making the loading on the truss zero and $s_n = 1$, or *vice versa* $\frac{ds}{ds_n} = u$ can be found as in Section 4, whichever way appears to offer the simplest solution.

SECTION 6.—TRUSS WITH ANY NUMBER OF SUPERFLUOUS BARS.—DEMONSTRATION OF THE THEOREM OF LEAST WORK.

Let us suppose now that we have a system of $(n-1)$ bars, numbered consecutively 1, 2, ...$(n-1)$, that we shall term "system N" of necessary bars, and in addition three superfluous bars numbered n, $(n+1)$, and $(n+2)$ respectively; so that the complete truss contains $n+2$ bars, of which the first $n-1$ are sufficient to fully define the form.

Let us use the notation of Section 5, extending it to the case of the extra superfluous bars by the use of the proper subscripts.

We shall give in this article, besides, the following meaning to the quantities below:

$V_1, V_2, V_3, \ldots\ldots \lambda_{n-1}$ = stresses, in bars 1, 2, ... $(n-1)$, of a frame supposed to consist of necessary bars alone (system N) subjected to the assumed loading.

$u_1, u_2, \ldots\ldots\ldots\ldots u_{n-1}$ = stresses in bars 1, 2, ...$(n-1)$ of "system N" alone, by forces unity acting toward each other from either end of the original position of superfluous bar n, the superfluous bars being supposed removed, however.

$V_1, V_2, \ldots\ldots\ldots\ldots V_{n-1}$ = stresses in bars 1, 2, ...$(n-1)$ of a frame supposed to consist of system N alone, caused by forces unity acting toward each other from the apices at the ends of superfluous bar $(n+1)$, all the superfluous bars being removed.

$r_1, r_2, \ldots\ldots\ldots\ldots r_{n-1}$ = same when forces unity act along original position of superfluous bar $(n+2)$.

The stresses Y, u, v and r can all be found by the laws of statics alone, without the aid of the theory of elasticity.

We shall further designate by,

$$\Delta a_n,\ \Delta a_{n+1},\ \Delta a_{n+2},$$

the amounts by which the apices at the ends of bars n, $(n+1)$, $(n+2)$ recede from each other under the actual final stresses and by,

$$\lambda_n,\ \lambda_{n+1},\ \lambda_{n+2},$$

the amounts by which bars n, $(n+1)$, and $(n+2)$ were too short to fit into the frame of necessary bars alone when under no stress, from temperature changes or actual misfit.

First consider the system with the superfluous bars removed and forces 1 acting toward each other along line of superfluous bar n.

Then by the principle of virtual velocities, as in section 4, we have, since $\Delta a_n + \lambda_n = c_n s_n$, or $\Delta a_n = -(\lambda_n - c_n s_n)$,

$$-u_1\,\Delta a_1 - u_2 \Delta a_2 - \ldots - u_{n-1}\,\Delta a_{n-1} + (\lambda_n - c_n s_n) = 0,$$

where Δa_1, Δa_2, ... $(c_n s_n)$, are the real changes of length of bars 1, 2, ...n in the complete structure under the final loading; for the real change of length $(c_n s_n)$ of the superfluous bar n is of a necessary consequence of the real changes of length of the necessary bars alone, and can therefore be found without knowing the changes of length in the other superfluous bars at all. In fact when the "system N" of necessary bars is built up from the final lengths of the necessary bars alone, the changes of length $c_n\ s_n$, $c_{n+1}\ s_{n+1}$, $c_{n+2}\ s_{n+2}$, and consequently the new lengths of the three superfluous bars, must correspond exactly to the corresponding new lengths, thus fully determined from the frame of necessary bars alone after stress.

If we next consider system N alone, acted on by forces 1, directed toward each other, along line of superfluous bar $(n+1)$, we have, since $c_{n+1}\ s_{n+1}$ is the stretch of bar $(n+1)$ (which must equal that deduced from a consideration of the frame of necessary bars alone after deformation), and $\Delta a_{n+1} = -(\lambda_{n+1} - c_{n+1}\ s_{n+1})$,

$$-v_1\,\Delta a_1 - v_2\,\Delta a_2 - \ldots - v_{n-1}\,\Delta a_{n-1} + (\lambda_{n+1} - c_{n+1}\ s_{n+1}) = 0.$$

Similarly, if we consider only system N acted on by forces 1 acting towards each other along line of superfluous bar $(n+2)$, we have,

$$-r_1\,\Delta a_1 - r_2\,\Delta a_2 - \ldots - r_{n-1}\,\Delta a_{n-1} + (\lambda_{n+2} - c_{n+2}\ s_{n+2}) = 0.$$

If we substitute for Δa_1 Δa_2 .... the following values, $\Delta a_1 = c_1 s_1$, $\Delta a_2 = c_2 s_2$, ..., we can write the above equations as follows:

$$u_1 c_1 s_1 + u_2 c_2 s_2 + \ldots + u_{n-1} c_{n-1} s_{n-1} + c_n s_n - \lambda_n = O. \ldots\ldots\ldots (1)$$
$$v_1 c_1 s_1 + v_2 c_2 s_2 + \ldots + v_{n-1} c_{n-1} s_{n-1} + c_{n+1} s_{n+1} - \lambda_{n+1} = O. . (2)$$
$$r_1 c_1 s_1 + r_2 c_2 s_2 + \ldots + r_{n-1} c_{n-1} s_{n-1} + c_{n+2} s_{n+2} - \lambda_{n+2} = O. . (3)$$

We shall now express the final stresses of the necessary bars as follows, as is evidently correct,

$$\left.\begin{aligned} s_1 &= Y_1 + u_1 s_n + v_1 s_{n+1} + r_1 s_{n+2} \\ s_2 &= Y_2 + u_2 s_n + v_2 s_{n+1} + r_2 s_{n+2} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ s_{n-1} &= Y_{n-1} + u_{n-1} s_n + v_{n-1} s_{n+1} + r_{n-1} s_{n+2} \end{aligned}\right\} \ldots\ldots (4)$$

On substituting these values in the three preceding equations, we have three equations between three unknown quantities s_n, s_{n+1}, s_{n+2}, so that the latter can be found, and on substituting their values in (4), we deduce the values of all the remaining stresses. In fact in any case, we always have as many equations like (1), (2), (3), as superfluous bars, so that it is always possible, no matter what the number of superfluous bars, to find the stresses in all the bars of the complete truss by following this general method.

This gives the solution by the method of virtual velocities. We shall now proceed to deduce the "theorem of least work" for this case.

Let us first note that Y_1, Y_2 .... Y_{n-1} in equations (4) are not functions of s_n, s_{n+1}, s_{n+2}, but that the last three stresses are functions of each other; since any change in any of them (due to varying the sections, say,) will change the stresses in all the other bars of the complete truss, just as equivalent external forces replacing the stresses in one bar, by being supposed to vary, can change the stresses in the remaining bars.

Therefore, if in taking the partial derivatives of s_1, s_2, ..., with respect to s_n, s_{n+1}, s_{n+2}, successively, we regard the stresses in all the superfluous bars as constant, except the one with respect to which the differentiation is effected, the operation is equivalent to considering these "constant" stresses as external forces, which thus are supposed to have the actual, final values of the stresses in the superfluous bars so treated, as constant.

With this understanding, then, we can write the following group of equations derived directly from (4) by differentiation:

$$\left.\begin{array}{llll} \dfrac{ds_1}{ds_n} = u_1, & \dfrac{ds_1}{ds_{n+1}} = v_1, & \dfrac{ds_1}{ds_{n+2}} = r_1 \\ \dfrac{ds_2}{ds_n} = u_2 & \dfrac{ds_2}{ds_{n+1}} = v_2, & \dfrac{ds_2}{ds_{n+2}} = r_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \\ \dfrac{ds_{n-1}}{ds_n} = u_{n-1}, & \dfrac{ds_{n-1}}{ds_{n+1}} = v_{n-1}, & \dfrac{ds_{n-1}}{ds_{n+2}} = r_{n-1} \end{array}\right\} \quad \ldots\ldots\ldots (5)$$

Now, substituting these values of u, v and r in equations (1), (2) and (3), we have,

$$c_1 s_1 \frac{ds_1}{ds_n} + c_2 s_2 \frac{ds_2}{ds_n} + \ldots\ldots + c_{n-1} s_{n-1} \frac{ds_{n-1}}{ds_n} + c_n s_n - \lambda_n = 0 \quad \ldots\ldots\ldots (6)$$

$$c_1 s_1 \frac{ds_1}{ds_{n+1}} + c_2 s_2 \frac{ds_2}{ds_{n+1}} + \ldots\ldots + c_{n-1} s_{n-1} \frac{ds_{n-1}}{ds_{n+1}} + c_{n+1} s_{n+1} - \lambda_{n+1} = 0 \quad \ldots\ldots\ldots (7)$$

$$c_1 s_1 \frac{ds_1}{ds_{n+2}} + c_2 s_2 \frac{ds_2}{ds_{n+2}} + \ldots\ldots + c_{n-1} s_{n-1} \frac{ds_{n-1}}{ds_{n+2}} + c_{n+2} s_{n+2} - \lambda_{n+2} = 0 \quad \ldots\ldots\ldots (8)$$

Now, if for convenience, we put the actual elastic work done by system "N" of necessary bars due to the final stresses equal to L,

$$\therefore L = \tfrac{1}{2} [c_1 s_1^2 + c_2 s_2^2 + \ldots\ldots + c_{n-1} s_{n-1}^2] \quad \ldots\ldots\ldots (9)$$

we see that equations (6), (7) and (8) can be derived by taking successively the partial derivatives, with respect to s_n, s_{n+1}, s_{n+2}, treated as independent of each other, of the expression,

$$L + \tfrac{1}{2} [c_n s^2_n + c_{n+1} s^2_{n+1} + c_{n+2} s^2_{n+2}] - [\lambda_n s_n + \lambda_{n+1} s_{n+1} + \lambda_{n+2} s_{n+2}] \ldots\ldots\ldots (10)$$

and placing them separately equal to zero.

In words, equation (6) shows us that if we treat the actual stresses in the superfluous bars $(n+1)$ and $(n+2)$ as constant external forces, the work of deformation of the remaining bars, diminished by the terms $\lambda_n s_n$, must be a minimum.

Equation (7) shows that the elastic work of "system N," with bar $(n+1)$ added, minus $\lambda_{n+1} s_{n+1}$, is to be a minimum, the stresses in the other two superfluous bars being treated as constant external forces. Similarly, equation (8) shows that the work of deformation of system N with bar $(n+2)$ added, minus $\lambda_{n+2} s_{n+2}$, is to be a minimum, the stresses in the other two superfluous bars being regarded as constant external forces, all the equations being agreeable to the laws of statics.

If λ_n, λ_{n+1}, λ_{n+2} the amounts by which the superfluous bars are too short, are zero, the results can be stated more succinctly as follows:

The stresses of the bars after deformation, are such as render a minimum the work of deformation of the system of necessary bars, combined with one of the superfluous bars at a time, the stresses in the rejected bars being treated as external forces, the results being always agreeable to the laws of statics.

We have been precise in stating this principle, for Castigliano, who deduces his conclusion in a totally different way, states it incorrectly as follows: "The tensions of the bars after deformation, are such as render a minimum the work of deformation of the system, having regard to the $3n-6$ equations between the tensions of the bars." The equations alluded to are those furnished by statics for each apex of the truss (not regarded as in one plane only), having regard to the conditions that must be fulfilled by the external forces.

It is to be inferred from this statement, which is repeated in several places, that Castigliano holds that the work of deformation of the entire truss must be a minimum, whereas it is only proved for the several partial trusses considered; and in fact in the applications given by that author to "compound systems," the rule or principle, as we have deduced it above, is followed to the letter. Further, the demonstration given by Castigliano leads to the same conclusion.

If we call F the elastic work of deformation of all the bars, including all the superfluous bars, we see that the expression (10) will be represented more simply by

$$F + \Sigma\lambda s,$$

extending the sum Σ only to the superfluous bars. The partial derivatives of this expression with respect to s_n, s_{n+1}, s_{n+2}, respectively placed equal to zero, give the equations (6), (7) and (8) needed; but in taking these partial derivatives s_n, s_{n+1}, s_{n+2}, must be regarded as independent of each other.

Similarly where the frame fits perfectly when not under strain, we have,

$$\frac{dF}{ds_n} = o, \frac{dF}{ds_{n+1}} = o, \frac{dF}{ds_{n+2}} = o$$

or the work of deformation of the whole system is a minimum with respect to s_n, s_{n+1}, s_{n+2} successively, provided we regard these stresses as independent in the differentiation. It gives a simpler rule to work

with when the work of the whole system is found and then differentiated successively as above, but we must not lose sight of the hypothesis that s_n, s_{n+1}, s_{n+2} are regarded as independent.

SECTION 7.—DERIVATIVE OF THE WORK OF DEFORMATION WITH RESPECT TO AN EXTERNAL FORCE.—DEFLECTION.

Consider a truss of invariable form, without superfluous bars, and let a force unity act in the direction of and along the line of action of any external force P. Then when the external forces are supposed removed, call u the stress in any bar due to the force unity in question. Let us suppose further that this bar elongates an amount $\Delta l = \frac{as}{e\omega}$, or the exact amount caused by the stress s due to the external forces when in place, and that this elongation alone causes the displacement Δp^1 of force unity in the direction of that force. Then by the principle of virtual velocities, we have, assuming u to be tension,

$$1.\ \Delta p^1 - u\Delta l = 0$$

$$\therefore \Delta p^1 = u\frac{as}{e\omega}$$

If u or s are compressive, they have the minus sign in the above equation, as is evident. Should Δp^1 thus become minus in any case, the displacement will be contrary to the direction of the supposed force unity. Continuing thus to find the displacement of force 1, due to the real change of length of each bar in turn, we have, for the total displacement of the point of application of force unity, acting in the direction of external force P, the known formula,

$$\Delta p = \sum\left(u\frac{as}{e\omega}\right)$$

But since this displacement is that caused by the actual stresses in all the bars due to the original external forces, it must equal the actual displacement of force P along its direction, or the deflection of the truss in the direction of force P.

Now calling X the stress in any bar due to all the loads and their corresponding reactions, when P is omitted, we have the stress in any bar,

$$s = X + uP$$

whence,

$$\frac{ds}{dP} = u,$$

since X is entirely independent of P.

$$\therefore \Delta p = \sum\left(\frac{as}{e\omega} \cdot \frac{ds}{dP}\right) = \frac{d}{dP}\left(\frac{1}{2}\sum\frac{as^2}{e\omega}\right) \ldots\ldots \text{(1)},$$

in which s must be replaced by $X + uP$.

We draw from this the important principle, of "the derivative of work," that *if we express the work of deformation of the bars as a function of the external forces, its derivative with respect to one of the forces gives the displacement, in the direction of the force of its point of application.*

Castigliano proves this principle in an entirely different manner and deduces, by its aid, very simply, the principle of least work.

If we call the work of deformation of the system F, it must be borne in mind in applying the principle that when $\frac{dF}{dP} = \Delta p$ is plus, the displacement is in the direction of the force; when minus in a contrary direction. The above demonstration refers only to an articulated frame without superfluous members, in which every external force, being independent, must be represented by a different letter.

Where two equal forces, directed both toward or both from each other along the same line, as in the case of the horizontal thrusts of an arch hinged at the ends at the abutments, are designated by the same letter P, if we call P and P^1 the two forces and F the work of deformation of the truss, then $\frac{dF}{dP}$ and $\frac{dF}{dP^1}$ give the actual displacements of P and P^1 resolved along the direction of the forces, both minus or both plus, according as the motion is opposed to the direction of the force or with it; so that $\frac{dF}{dP} + \frac{dF}{dP^1}$ gives the total relative displacements of P and P^1. Similarly if the superfluous members in any truss are each supposed replaced by two equal forces at either end, each equal to the final stress in the member and designated by the same letter, say P for one of the superfluous bars, then if F represents the work of deformation of the necessary bars, expressed as a function of all the external forces, including there the forces replacing the stresses in the superfluous bars, then as before $\frac{dF}{dP} + \frac{dF}{dP^1}$ gives the total relative displacement of the apices to which the forces P and P^1 $(= P)$ are applied. Now if we regard P^1 as a function of P, the total derivative of F with respect to P, is

$$\frac{dF}{dP} + \frac{dF}{dP^1}\frac{dP^1}{dP};$$

but since $P^1 = P, \frac{dP^1}{dP} = 1$, therefore the total derivative of F with respect to P is $\frac{dF}{dP} + \frac{dF}{dP^1}$, or the total relative displacement of the apics where P and P^1 are applied. Consequently with this interpretation of results, it is not necessary, in the cases supposed, to designate the two equal opposed forces, by different letters.

When any apex of the truss is subjected to any known displacement Δp, the above formula enables us to compute the value of the force P at the apex when statics leaves it undetermined. Thus, if a truss with no superfluous bars, but not resting on rollers, is subjected to the condition that the yielding of the abutments shall be a certain amount Δp, the horizontal component of the reactions, P, is found by solving the preceding equation $\frac{dF}{dP} = \Delta p$, in which we express the value of the stress in any bar as a function of P as before. As a particular example take the case of

The Braced Arch Hinged at the Springing.

We shall suppose that the truss proper has no superfluous bars, but that it is subjected to the condition that the span is invariable. Call $P = H =$ horizontal component of the thrust at the abutment, the corresponding displacement of H being $\Delta p = 0$.

Then calling, as before, $X =$ stress in a bar due to external loads and vertical reactions resulting, H being omitted, we have from equation (1), on substituting the value $s = X + u\,H$,

$$\sum \left(u \frac{aX}{e\omega} + a \frac{u^2 H}{e\omega} \right) = 0.$$

The sum Σ being extended to all the bars. In this equation u represents the stress caused in any bar by a force unity at the left abutment, say, acting inward, or with the direction of H, horizontally toward the corresponding reaction unity, acting horizontally inward, at the other abutment. From this equation, in which H is the only unknown quantity, we can at once find the value of H and then the stress in any bar from the equation, $s = X + uH$.

Another method of procedure has already been illustrated for this case of span invariable, in example 1 of Section 5, by simply considering the work in a supposed superfluous horizontal bar to become zero. We get in this way the stress in some bar of the truss from the resulting

equation, and afterward the horizontal thrust and other stresses as readily as by this method.

Braced Arch Fixed at Abutments and Hinged at Middle.

Here we have two unknown forces to determine, the vertical component P and the horizontal component H of the thrust at the crown, or the action of either half of the arch upon the other half. Suppose the thrust of the right half of the arch upon the left half to be directed to the left and upward, so that P is positive upward and H positive to the left in considering the action of the right half upon the left. Call s the actual stress in a bar of the left half due to the final loading, v = stress from force 1 alone at crown acting downward and opposed to direction P, u = stress from force 1 alone acting to right at crown opposed to direction of H, and X the stress caused in any bar of the left half by the actual loading, when P and H are zero, or no thrust at the crown. Designate by s_1, u_1, v_1, X_1, the same quantities for the right half where P and vertical force 1 act downward, and H and horizontal force 1 act to right, at the crown,

$$\therefore s = X - uH - vP.$$
$$s_1 = X_1 + u_1 H + v_1 P.$$

Now, since the vertical and horizontal displacements, downward and to the right respectively, at the crown from the final stresses of the two halves must be separately equal, we have, from the principle above,

$$-\sum\left(\frac{as}{e\omega}\cdot\frac{ds}{dP}\right)=\sum\left(\frac{a_1 s_1}{e_1\omega_1}\cdot\frac{ds_1}{dP}\right)$$
$$-\sum\left(\frac{as}{a\omega}\cdot\frac{ds}{dH}\right)=\sum\left(\frac{a_1 s_1}{e_1\omega_1}\cdot\frac{ds_1}{dH}\right)$$

in which s and s_1 have the values given above, and the sum Σ in the left members extends to all the bars in the left half, and the sum Σ in the right members extends to all the bars in the right half, of the arch.

These equations can otherwise be written,

$$\sum\left(\frac{as}{e\omega}v\right)=\sum\left(\frac{a_1 s_1}{e_1\omega_1}v_1\right)$$
$$\sum\left(\frac{as}{e\omega}u\right)=\sum\left(\frac{a_1 s_1}{e_1\omega_1}u_1\right)$$

From these two equations, after substituting the values of s and s_1 above, can be found the two unknowns H and P, from which the values of all the stresses can be found from the equations above.

Braced Arch without Hinges.

The braced arch without hinges, and with but one system of triangulation, is shown in Fig. 4. It is plain that with immovable abutments the figure of this arch, *i. e.*, all the apices, can be completely defined by leaving out three bars, say AB, AC and CD; for, on working from the right abutment and fixing each apex by two sides from two apices already established, we finally determine the position of apices B and C, so that with the immovable points A and D of the left abutment, the position of the truss is completely determined without knowing the lengths of the aides AB, AC and CD, which may thus be regarded as superfluous in defining the form of the truss.

Moreover the lengths of the bars so named must exactly fit the distances between the apices B and C so determined and the fixed points A and D at the abutments.

Designate the superfluous bars AB, AC and CD as bars n, $n+1$ and $n+2$ respectively, their actual stresses due to loading by s_n, s_{n+1}, s_{n+2} and their lengths, moduli and cross-sections, by a, e and ω with the corresponding subscripts.

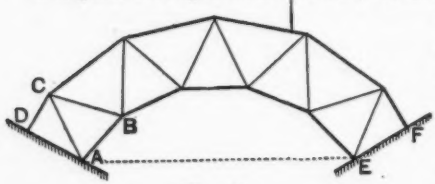

Fig. 4.

We shall designate the system composed of the remaining bars as "system N" of necessary bars, as they can alone define the form of the truss. The stress, length, modulus and cross-section of one of the necessary bars will be designated generally by the letters s, a, e and ω, the sums Σ below will refer to the necessary bars alone.

Let us call Y, u, v and r the stresses caused in a bar of the cantilever $EFCB$ of necessary bars alone, respectively by the loading, force 1 acting from B toward A, force 1 acting from C toward A, and force 1 acting from C toward D; which stresses can be found by the laws of statics alone. Then the actual stress in any bar will be given generally by the expression,

$$s = Y + u\,s_n + v\,s_{n+1} + r\,s_{n+2}$$

The work of deformation of the necessary bars alone is,

$$\frac{1}{2}\sum\frac{as^2}{e\omega},$$

in which s has the value just given.

Now, regarding s_n, s_{n+1}, s_{n+2} as constant exterior forces, we have by the principle of the derivative of work above, that the diffential co-efficients of $\frac{1}{2}\sum\frac{as^2}{e\omega}$ with respect to s_n, s_{n+1} and s_{n+2}, successively, will give the displacements of point B in the direction BA, of point C along CA, and of C toward D, respectively. Now as these displacements, added to the changes of length of bars AB, CA and CD, respectively, must equal zero, since the points A and D are immovable, we must have separately,

$$\sum\left(\frac{as}{e\omega}\cdot\frac{ds}{ds_n}\right)+\frac{a_n\,s_n}{e_n\,\omega_n}=0.$$

$$\sum\left(\frac{as}{e\omega}\cdot\frac{ds}{ds_{n+1}}\right)+\frac{a_{n+1}\,s_{n+1}}{e_{n+1}\,\omega_{n+1}}=0.$$

$$\sum\left(\frac{as}{e\omega}\cdot\frac{ds}{ds_{n+2}}\right)+\frac{a_{n+2}\,s_{n+2}}{e_{n+2}\,\omega_{n+2}}=0.$$

If the Σ in the first equation is positive, indicating a shortening of the distance BA, then s_n is negative or compression. Similarly for the others. On substituting the values of s given above, also the values of $\frac{ds}{ds_n}=u$, $\frac{ds}{ds_{n+1}}=r$, $\frac{ds}{ds_{n+2}}=v$, we have three equations between the three unknown quantities s_n, s_{n+1}, s_{n+2}, so that the latter can be found, from which the values of all the remaining stresses can be determined from expressions of the type, $s=Y+u\,s_n+vs_{n+1}+rs_{n+2}$.

Section 8.—Second General Demonstration of the Theorem of Least Work.

A demonstration of the theorem of least work can likewise be given for the case of any number of superfluous bars, by aid of the theory of deflection just demonstrated.

Thus, employing the notation used in the last example, for a truss of any kind with any number of superfluous bars, numbered n, $n+1$, $n+2$, ..., the total elastic work of deformation of all the bars, including the superfluous bars, is,

$$F=\frac{1}{2}\sum\left(\frac{as^2}{e\omega}\right)+\tfrac{1}{2}\frac{a_n\,s_n^{\,2}}{e_n\,\omega_n}+\tfrac{1}{2}\frac{a_{n+}a_{n+1}\,s_{n+1}^{\,2}}{e_{n+1}\,\omega_{n+1}}+\ldots.,$$

in which the sum Σ extends to the necessary bars alone. It is understood, in this and subsequent expressions, that for every value of the stress s in a necessary bar we must substitute expressions of the type given above,

$$s = Y + us_n + vs_{n+1} + \ldots,$$

so that F will be expressed as a function of s_n, s_{n+1}, $\ldots$.

Now, designating by L the work of deformation of the necessary bars alone, we have,

$$L = \frac{1}{2} \sum \left(\frac{as^2}{e\omega} \right),$$

in which L must likewise be expressed as a function of s_n, s_{n+1}, $\ldots$, as before.

Now, consider the superfluous bars n, $n+1$, $\ldots$, temporarily removed and replace their action by two forces for each bar, each equal to the stress in the superfluous bar and acting toward each other, as we assume, in framing out equations, that all bars are in tension.

It has been shown above that treating these forces s_n, s_{n+1}, $\ldots$, as external forces and independent of each other, that $-\frac{dL}{ds_n}$ represents the increase in distance between the apices at the extremity of bar n, the minus sign being used since the two forces s_n, s_n replacing the tension of the bar, act in the opposite direction to the displacements. Similarly for the other deflections.

Now, since s_n is supposed to equal the actual stress in bar n in the complete structure under the loading, it follows that $-\frac{dL}{ds_n}$, which represents the exact increase of distance between the apices, must equal the elongation of the bar n under the stress s_n when in place; therefore, we must have,

$$-\frac{dL}{ds_n} = \frac{a_n s_n}{e_n \omega_n},$$

or,

$$\frac{dL}{ds_n} + \frac{a_n s_n}{e_n \omega_n} = O$$

We should have a similar expression for each of the superfluous bars, or as many equations as unknown quantities.

Now, each equation of the type just shown can be found by taking the partial derivatives of the expression for F given above, with respect to s_n, s_{n+1}, $\ldots$, treated as independent of each other, successively, and placing the results equal to zero. In other words, to determine the unknown stresses we express the work of deformation of the whole system as a function of the stresses in the bars taken as superfluous, then treating these stresses as independent of each other in the differentiation, we express that the work of the necessary bars and one superfluous

bar at a time be a minimum. It is this which constitutes what is called the method of least work.

SECTION 9.—EXTENSION OF THE THEOREMS OF DEFLECTION AND LEAST WORK TO SOLID BEAMS AND TO STRUCTURES WHERE THE PIECES ARE NOT ALL FREE TO MOVE AT THE JOINTS.

In treating of a solid beam, straight or curved, which may of itself constitute a sufficient structure either alone or when in combination with other pieces, we shall assume with Lamé, St. Venant and others, that solids are composed of very small molecules, separated by very small intervals. Further, we shall assume as the result of experience, that—

When the distance between two molecules has been changed by a quantity very small in relation to the distance between their centers, from the application of exterior forces, there is developed along the line of these centers, an elastic force proportional to the change of length of this distance, which is attractive or repulsive according as the two molecules are made to recede from or approach each other.

The law being admitted, we see that all the bodies and all elastic systems whatsoever can be regarded as articulated systems, and consequently enjoy the properties of such systems already demonstrated, for the hypothesis is identical with the law of elasticity first assumed in Section 1.

Hence, from the reasoning of Section 7, we have for all kinds of elastic systems, the following theorem for ascertaining deflection:

If we express the work of deformation of a body or an elastic system as a function of the exterior forces, including the stresses in the superfluous bars, (if any) as independent exterior forces, the derivative of this expression in relation to any one of these forces, gives the displacement in the direction of the force of its point of application.

When this derivative is plus, the displacement is in the direction of the force; when minus, in the opposite direction.

From the reasoning of Sections 6 and 8, we finally deduce the theorem of least work for any kind of an elastic system:

The elastic forces experienced between the molecules after deformation correspond to a minimum of the work of deformation of the system, expressed as a function of certain stresses, taken with respect to these stresses successively, regarded as independent during the differentiation.

The first step in applying this principle is always the most laborious,

viz.: finding F, the work of deformation of the entire system in terms of certain stresses s_n, s_{n+1}, ...; the subsequent work consisting simply in placing the partial derivatives of F with respect to s_n, s_{n+1}, ..., regarded as independent variables for the nonce, equal to zero, thus giving the equations

$$\frac{dF}{ds_n} = O, \quad \frac{dF}{ds_{n+1}} = O, \quad \ldots,$$

from which the values of s_n, s_{n+1}...., can at once be found by elimination, and subsequently the molecular stresses anywhere in the system.

Now it often happens that it is more convenient to express the work of deformation in terms of certain resultants and moments of the molecular stresses, acting along certain interior plane faces, than to express this work in terms of certain elementary molecular stresses as assumed. But we shall find by the aid of the following investigation that the above theorems hold when we substitute the new unknown quantities for those first assumed.

Section 10.—Change of the Unknown Quantities.

Suppose that the work of deformation of an elastic system may be expressed as a function of certain unknown stresses P, Q, R...., which, according to the theorem of least work, should be determined by placing the partial derivatives of the work of deformation in relation to these quantities equal to zero.

Now when it happens, as it does very often, that these forces can be expressed as a function of an equal number of other unknown quantities, p, q. r ...., connected to the first by equations of the first degree, we can express the relationship in the most general manner, as follows:

$$\left.\begin{aligned} P &= a_1 p + b_1 q + c_1 r + \ldots, \\ Q &= a_2 p + b_2 q + c_2 r + \ldots, \\ R &= a_3 p + b_3 q + c_3 r + \ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\}$$

a_1, b_1, c_1, ..., a_2, b_2, c_2, ..., a_3, b_3, c_3, ..., being co-efficients independent of the unknown quantities. We propose now to demonstrate the following very important property:

If we express the work of deformation of the system as a function of the new unknown quantities, p, q, r, ..., these can be found directly by equating to zero the derivatives of the work of deformation in relation to these same quantities.

In fact, if we call L the work of deformation of the system, we can regard L as a direct function of the unknown stresses P, Q, R, ..., and these as functions of the new unknown quantities p, q, r, ..., from the equations just given.

We have then, by known rules of the differential calculus;

$$\frac{dL}{dp} = \frac{dL}{dp}\frac{dP}{dp} + \frac{dL}{dQ}\frac{dQ}{dp} + \frac{dL}{dR}\frac{dR}{dp} + \ldots.;$$

then on substituting here for $\frac{dP}{dp}, \frac{dQ}{dp}, \frac{dR}{dp}$, ..., their values a_1, a_2, a_3, ..., we obtain the first of the following relations, and the others are similarly found.

$$\frac{dL}{dp} = a_1\frac{dL}{dP} + a_2\frac{dL}{dQ} + a_3\frac{dL}{dR} + \ldots.,$$

$$\frac{dL}{dq} = b_1\frac{dL}{dP} + b_2\frac{DL}{DQ} + b_3\frac{DL}{dR} + \ldots.,$$

$$\frac{dL}{dr} = c_1\frac{dL}{dP} + c_2\frac{dL}{dQ} + c_3\frac{dL}{dR} + \ldots.,$$

.....

But since we have by hypothesis,

$$\frac{dL}{dP} = O, \frac{dL}{dQ} = O, \frac{dL}{dR} = O \ldots.;$$

we deduce as a consequence from the above equations

$$\frac{dL}{dp} = O, \frac{dL}{dq} = O, \frac{dL}{dr} = O \ldots.;$$

which demonstrates the theorem above.

Section 11.

We shall now demonstrate the following theorem: *The differential coefficient of the work of deformation of an elastic system with respect to the moment of a couple composed of two external forces, perpendicular to the straight line which joins their points of application, expresses the rotation of this line about an axis perpendicular to the plane of the couple.*

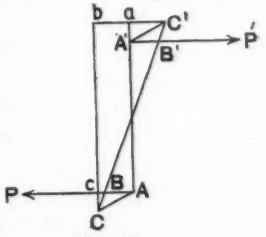

Fig. 5

In Fig. 5, the two equal but opposite parallel external forces P and P^1, both perpendicular to the straight line AA^1 joining their points of application, form a couple whose moment we shall call M. P and P^1 in this case represent "external forces," acting on a structure or forces

replacing the stresses in certain bars, which will be treated exactly as if they were independent external forces in any case.

We have,

$$P.\,\overline{AA^1} = M,\ P^1.\,\overline{AA^1} = M.$$

Now, we can regard the work of deformation of the whole system of necessary bars as a function of the variables P, P^1, which in their turn will be regarded as functions of M, so that the derivative of F in relation to this last quantity will be,

$$\frac{dF}{dM} = \frac{dF}{dP}\cdot\frac{dP}{dM} + \frac{dF}{dP^1}\frac{dP^1}{dM}.$$

But since,

$$\frac{dP}{dM} = \frac{1}{AA^1},\ \frac{dP^1}{dM} = \frac{1}{AA^1},$$

we have,

$$\frac{dF}{dM} = \frac{1}{AA^1}\left(\frac{dF}{dP} + \frac{dF}{dP^1}\right).$$

Now, by section 7, $\frac{dF}{dP}$ and $\frac{dF}{dP^1}$ give the projections of the displacements, along the line of the forces P and P^1 respectively, of the points of application of these forces.

If these displacements are directly along the directions of the forces, as AB and A^1B^1, Fig. 5, then it is evident that, for this case,

$$\frac{dF}{dM} = \frac{\overline{AB} + \overline{A^1B^1}}{\overline{AA^1}}$$

expresses the rotation of the line AA^1 about an axis perpendicular to it and the theorem is proved.

For any case of displacement of the points A and A^1 let C, C^1 be the projections of the new positions of A and A^1 on the plane of the couple. It is evident that the rotation of $\overline{AA^1}$ about an axis perpendicular to the plane of the couple is expressed by the angle between the lines AA^1 and CC^1. Now, if from the point C we draw the straight line Cb parallel to AA^1, and from the point C^1 the line C^1b perpendicular to AA^1 and if we prolong the straight line AA^1 to a, we see that C^1a and ab will be the projections of the displacements A^1C^1, AC upon the direction of the forces, and will have for values $\frac{dF}{dP}$, $\frac{dF}{dP^1}$; both plus (Section 7) when these projections fall upon the side of the forces, both minus when they fall on the side of their prolongations.

The angle C^1Cb has a tangent whose value is

$$\frac{C^1b}{Cb} \text{ or } \frac{C^1b}{AA^1}$$

on neglecting the quantities A^1a, Cc, as infinitely small in relation to AA^1.

As the angle C^1Cb is very small, we can replace it by its tangent, whence

$$\frac{dF}{dM} = \frac{1}{AA^1}\left(\frac{dF}{dP} + \frac{dF}{dP^1}\right) = \frac{C^1a + ab}{AA^1} = \frac{C^1b}{AA^1}$$

which demonstrates the theorem since $\frac{C^1b}{AA^1}$ gives the rotation of AA^1 around an axis perpendicular to the plane of the couple.

As the partial derivatives $\frac{dF}{dP}, \frac{dF}{dP^1}$ are both plus or both minus, according as the projections of the displacements AC, A^1C^1 fall upon the direction of the forces or upon their prolongations, it follows that

$$\frac{dF}{dM} = \frac{1}{AA^1}\left(\frac{dF}{dP} + \frac{dF}{dP^1}\right)$$

will be plus or minus, according as the rotation of the straight line AA^1 is in the direction of the action of the couple or in the contrary direction.

Where P^1 and P represent the resultants of forces as in a beam, uniformly increasing from a neutral axis, as in Fig. 6 (which can be supposed to represent the action of the beam to right of section on part to left), and a plane section AB of the beam before strain, is plane after strain, the same principle holds, and it is to be observed that the rotation of the couple PP^1 is the same as that of the line AB perpendicular to the forces, representing the rotation of the elementary forces two and two, or of the plane section AB after strain.

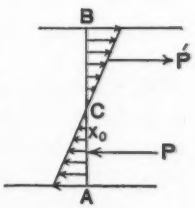

Fig. 6

If this is not evident, call $\alpha =$ the angle of rotation of section AB about C, call p any elementary force below C, a distance x, and let x_o be the distance of P below C and α_o the rotation of couple PP^1, formed of the two resultants P and P^1, then by the principle of virtual velocities, since P is the resultant of Σp,

$$P\,(x_o\,\alpha_o) = \Sigma\,(p.x\alpha) = \alpha\,\Sigma\,(p.x)$$

the sum Σ extending from C downward. But since by the principle of moments,

$$Px_o = \Sigma\,(px)$$

it follows that $\alpha_o = \alpha$, which was to be proved. Hence in this case, if M represents the moment of the couple PP^1, $\frac{dF}{dM}$ gives the rotation of a plane section AB (Fig. 6) after strain. It must be carefully borne in mind that the forces P and P^1 in question must be treated as independent external forces, for the demonstration of Section 7 shows that $\frac{dF}{dP}$ represents the deflection along P of the apex to which it is applied, only when P is treated as one of the external forces acting on a truss without superfluous members; similarly for P^1.

It is important to note in both the cases examined that if the structure has superfluous members, F can still represent the elastic work of deformation of the entire truss; only, as was shown in Section 7 in taking the partial derivatives, $\frac{dF}{dP}$, $\frac{dF}{dP^1}$, the forces P and P^1 (which generally replace certain stresses) must be regarded as independent of each other and of the stresses in the superfluous bars, which are conceived to be replaced by constant forces having the values of the final stresses in those bars (thus giving a truss without superfluous bars, to which the theory of deflection applies), so that the terms involving these stresses disappear in the differentiation.

Thus to express that the rotation of a certain couple in a structure is α, we find an expression for F the work of deformation of the whole truss in terms of M and certain stresses, T, S ...., in superfluous members. Then although the value of M may depend upon the values of T, S ...., the rule deduced for rotation requires that we differentiate F with respect to the letter M, treating T and S ...., as constants and then state the equality,

$$\frac{dF}{dM} = \alpha.$$

We have been precise in stating the method of dealing with F, the whole work of deformation, for the different cases, as Castigliano has left it in considerable doubt until the applications are reached. The method of virtual velocities, that we have used in demonstrating the principles in question, is so precise as to cause no hesitancy in the applications.

Section 12.—Work of Deformation.

The work of deformation of a tie or strut, is given by equation (3) of Section 1. When the longitudinal stress is uniform at each cross-section but varying with the length of piece, call P the longitudinal stress at any point, normal to the cross-section of area A, and regard it as constant for the distance Δx along the axis of a prismatic bar; then for a gradually applied load the work of deformation for the length Δx is nearly $\frac{P^2 \Delta x}{2EA}$, and for a beam of length l exactly,

$$\frac{1}{2EA}\int_0^l P^2 dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly, if S is the shear at any point, then at a distance dx along the axis, we have another equal but opposite shear S which moves, relatively to the first, a distance $\alpha\ dx$, if α is the angle (in arc) through which the part dx of the axis is turned; then calling F the modulus of shearing elasticity, we have the known equation,

$$S = \alpha F A$$

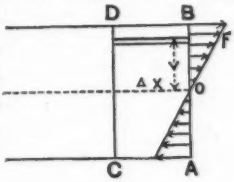

Fig. 7

if the shear is regarded as uniformly distributed over the cross-section, and if the load is applied gradually, the work of deformation for the length dx is,

$$\frac{S}{2}\alpha dx = \frac{S^2}{2FA}dx;$$

and for the entire length,

$$\frac{1}{2FA}\int_0^l S^2 dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Work of Deformation Due to Flexure.

We shall assume the well-known formula for the unit stress f at a distance v from the axis of flexure, when acted on by a couple whose moment about this axis is called M, the corresponding moment of inertia of the section being I,

$$f = \frac{Mv}{I} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

In Fig. 7, representing a part of a beam, the axis of flexure for the section AB is taken at O and at right angles to the plane of the paper. The axis of Z will be supposed to coincide with the axis of

flexure, so that the moment of inertia of the section AB about the axis of flexure $= \int\int v^2 dvdz$.

Now, if we consider another cross-section CD, Δx from the first, we can conceive the stress f on a fiber of cross-section unity, to be nearly uniform for the very small distance Δx, so that the work of deformation of this fiber contained between the two cross-sections, by equation (3) of Section 1 is for a gradually applied load,

$$\frac{1}{2}\frac{f^2 \Delta x}{E} = \frac{1}{2}\frac{M^2 v^2}{EI^2}\Delta x.$$

If the fiber is regarded as having a cross section $\Delta v \Delta z$, the work of deformation is nearly,

$$\frac{1}{2}\frac{M^2 v^2}{EI^2}\Delta x \Delta v \Delta z$$

Hence the total work of deformation on an elementary slice of the solid Δx long is nearly,

$$\frac{1}{2}\Delta x \int\int \frac{M^2}{EI^2} v^2 dvdz = \Delta x \frac{M^2}{2EI}.$$

We have only to take the limit of the sum of such values for the whole length l of the prism, as Δx diminishes indefinitely, to get the exact expression for the work of deformation due to flexure,

$$\frac{1}{2EI}\int_o^l M^2 dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

If the terms A and I are variable, insert to right of integration sign in (1) to (4).

WORK OF DEFORMATION OF AN ARCHED BEAM CONTINUOUSLY LOADED, WHEN ITS AXIS IS ANY CURVE WHATSOEVER AND THE CROSS-SECTION IS VARIABLE.

Divide the axis of the solid into an even number n of parts, each of the same length l, and so small that they can be regarded without sensible error as straight, and number the sections, beginning at one end, 0, 1, 2. 3, $\ldots . n$, continuously. Then designate the moments of flexure about horizontal axes passing through the centers of gravity of the sections M_o, M_1, $M_2 \ldots .$, M_n respectively; the normal stresses P_o, P_1, $P_2 \ldots$ P_n and the corresponding shearing forces, S_o, S_1, $S_2 \ldots .$, S_n, the subscript in each case referring to the section so numbered. The total work of deformation is expressed by the formula,

$$\frac{1}{2E}\int\frac{M^2}{I}\,ds+\frac{1}{2E}\int\frac{P^2}{A}\,ds+\frac{1}{2F}\int\frac{S^2}{A}\,ds,$$

the integrations covering the whole length of the solid.

Now, if we describe a curve, taking for abscissas the values of the arc s (the origin being taken at one end) and for ordinates the corresponding values of $\frac{M^2}{I}$, it is clear that the area between this curve, the axis of abscissas and the extreme ordinates will be represented by the integral, $\int\frac{M^2}{I}\,ds$.

But according to Simpson's formula we can express approximately this area in the following manner: we divide the portion of the axis of abscissas comprised between the extreme ordinates into an even number n of parts, each of length l, designate the ordinates at the successive points of division, $y_o, y_1, y_2, \ldots y_n$, then the area sought is equal to,

$$\frac{l}{3}(y_o+4y_1+2y_2+4y_3+\ldots+4y_{n-1}+y_n)$$

Therefore we have,

$$\int\frac{M^2}{I}\,ds=\frac{l}{3}\Big(\frac{M_o^2}{I_o}+4\frac{M_1^2}{I_1}+2\frac{M_2^2}{I_2}+4\frac{M_3^2}{I_3}+\ldots+4\frac{M^2{}_{n-1}}{I_{n-1}}+\frac{M_n^2}{I_n}\Big)$$

Applying the same reasoning to the other terms, we have the work of deformation of the entire solid equal to,

$$\left.\begin{aligned}&\frac{1}{2E}\frac{l}{3}\Big(\frac{M_o^2}{I_o}+4\frac{M_1^2}{I_1}+2\frac{M_2^2}{I_2}+4\frac{M_3^2}{I_3}+\ldots+4\frac{M^2{}_{n-1}}{I_{n-1}}+\frac{M^2{}_n}{I_n}\Big)\\&+\frac{1}{2E}\frac{l}{3}\Big(\frac{P_o^2}{A_o}+4\frac{P_1^2}{A_1}+2\frac{P_2^2}{A_2}+4\frac{P_3^2}{A_3}+\ldots+4\frac{P^2{}_{n-1}}{A_{n+1}}+\frac{P_n^2}{A_n}\Big)\\&+\frac{1}{2F}\frac{l}{3}\Big(\frac{S_o^2}{A}+4\frac{S_1^2}{A_1}+2\frac{S_2^2}{A_2}+4\frac{S_3^2}{A_3}+\ldots+4\frac{S^2{}_{n-1}}{A_{n-1}}+\frac{S_n^2}{A_n}\Big)\end{aligned}\right\}\quad\ldots\ldots\ldots(5)$$

If the quantities M, P, S, I, A are not continuous functions of the arc s (in which case alone Simpson's formula is applicable) we must divide the solid into such portions as are continuous, taking as points of

division those where the changes are sudden, and afterward apply (5) to each of these parts and add the results.

The above formula is equally true for a straight beam, and it is interesting to note in the last case that when the section of the solid is regarded as constant, the formula is exact, as can easily be proved. As it stands, the formula is practically exact, and affords a means of investigating the strength and stability of the arch of variable cross-section, which ordinary analytical methods leave untouched.

For special cases of straight beams, the formula can be modified slightly with advantage, but we shall not stop to consider further the applications to straight beams, as ordinary methods suffice; but it may be remarked that the method of work leads to the well known formulas for the strength and deflections of beams, and some of the solutions are quite direct and elegant.

Section 13.—Solid Arch Fixed at the Ends.

The problem is to find the resultant pressure at the crown in position, direction and magnitude, having given the dimensions of the arch and the loading.

The resultant on any normal section of the arch is generally inclined to the normal to the section, and may not pass through the center of gravity of the section. It can therefore be decomposed into a normal pressure P and shearing force S. Now if we conceive two opposed forces each equal and parallel to P to act at the center of gravity of the cross-section, we do not disturb equilibrium; but we have the one force P giving a uniform compression $\frac{P}{\omega}$ over the cross-section, calling ω its area, and the other combined with the original component P acting at a distance d from the center of gravity, giving a couple whose moment is $M = Pd$, which causes a stress uniformly increasing from the center of gravity, tension on one side, compression on the other, whose amount is $\frac{Mv^1}{I}$, where v^1 is the distance of the fiber experiencing this stress from the neutral axis and I, the moment of inertia of the cross-section about a horizontal axis passing through its center of gravity. The maximum fiber stress is experienced at the intrados or extrados, so that if v represents the distance from the center of gravity of the cross-section to either curve, we have the maximum unit fiber stress given by the formula,

$$\frac{P}{\omega} \pm \frac{Mv}{I}$$

Similarly at the crown joint, we can decompose the single resultant there into a normal thrust (p) acting at the center of gravity of the joint, a shear (s) acting along the section, and a couple (p,p) whose moment is $m = pd$.

Then by the method of the derivative of work, if we suppose one-half of the arch removed and its action at the crown to be replaced by the forces p and s, which, regarded as external forces, will hold the other half in equilibrium; the derivative of the elastic work of deformation of the half arch with respect to either of these forces will give the corresponding displacement of the crown in the direction of the force (Section 9), and by Section 11, the derivative with respect to the moment m, gives the rotation of the plane section where it is applied.

Thus if L' and L'' represent respectively the work of deformation of the left and right halves of the solid arch considered, $\frac{dL'}{pp}$ gives the horizontal displacement of the point of application of p at the crown of the left half and $\frac{dL''}{dp}$ for the right half; so that $\frac{dL'}{dp} + \frac{dL''}{dp} = \frac{dL}{dp} = O$, where $L = L' + L''$; for these displacements are necessarily equal but of opposite signs, since p is directed to the left in one case and to the right in the other. Similarly the vertical displacements are equal and of contrary sign, so that,

$$\frac{dL'}{ds} + \frac{dL''}{ds} = \frac{dL}{ds} = O.$$

Lastly, the rotation of the section of the crown for the semi-arch to the left must be equal and of contrary sign to the rotation of the same section for the semi-arch to the right, so that,

$$\frac{dL'}{dm} + \frac{dL''}{dm} = \frac{dL}{dm} = O.$$

The same results can be deduced by the principle of least work of Section 9. Thus it is easily shown that the complete stress between any two molecules can be expressed as a linear function of p, s and m; therefore we can deduce the values of p, s and m, and hence of L, in terms of the stresses t_1, t_2, t_3, at any three points, which may be regarded as the three unknown stresses to be found by placing (Section 9),

$$\frac{dL}{dt_1}=O,\ \frac{dL}{dt_2}=O,\ \frac{dL}{dt_3}=O;$$

whence by Section 10, since t_1, t_2, t_3, can be expressed as linear functions of p, s, m, we can find the latter at once by placing,

$$\frac{dL}{dp}=O,\ \frac{dL}{ds}=O,\ \frac{dL}{dm}=O,$$

as found by the method of deflections.

The tedious part of this solution is in finding L. To do this, we proceed, as in Section 12, for the arch of variable cross-section, to divide the line passing through the centers of gravity of all the cross-sections of the arch ring into an even number n of equal parts, of length l, and denoting the normal pressure, shear and moment at any cross-section by the letters p, s and m with the subscript pertaining to that cross-section, we have the total work of deformation given by formula (5) of Section 12, in which, however, the terms pertaining to s can be omitted, as producing no sensible influence upon the result. In fact for a first approximation, the terms involving p, giving the influence of the uniform compression, can be omitted.

It is evident that the values of $m_1\ p_1\ s_1$, $m_2\ p_2\ s_2 \ldots$, for any section, can be expressed as linear functions of m, p and s for any kind of loading, so that L will eventually involve only terms of the first and second degrees as to m, p and s; and when we differentiate L successively with respect to m, p and s, and place the results equal to zero, we have three equations of the first degree with respect to m, p and s, from which they can be found by elimination, and ultimately m_1, p_1, s_1, ..., and the maximum fibre stresses at any of the cross-sections by a formula already given. Also, since at any cross-section as 2, $m_2 = p_2\ d_2$, we have the distance of the center of pressure on that joint from its center of gravity given by the formula $d_2 = \frac{m_2}{p_2}$ from which the line of pressures can be drawn; or if preferred, p and s can be combined at the crown, at the distance $d = \frac{m}{p}$ from the center of gravity of the crown joint, thus giving the desired resultant pressure at the crown, from which the line of pressures can be located by the usual graphical construction.

Stresses Due to Temperature.

If the arch fits without strain to the span and abutments at the temperature t, then at the temperature t^1, if free, it would have a similar

shape to its first figure, so that the angles of the extreme sections would be exactly the same in the two cases; but the distance between the centers of these sections, originally l, would be increased by $\alpha l\,(t^1 - t)$ where α is the increase of length of the material of the arch for a length unity, and a rise of 1 degree of temperature. Now, since the span and skew-backs are invariable, the crown joint of the left half of the arch, if free to expand by itself, would move to the right, and that of the right half if free, to the left—distances whose sum $= \alpha l\,(t^1 - t)$. But the new thrust at the crown would move this joint to the left, if acting only on the left half of the expanded half arch, a distance $-\frac{dL^1}{dp}$, and the crown joint of the right half, if free, to the right, a distance $\frac{dL''}{dp}$, so that the total

$$\frac{dL^1}{dp} + \frac{dL''}{dp} = \frac{dL}{dp},$$

must exactly equal $\alpha l\,(t^1 - t)$, in order that the crown joints of the two halves may exactly coincide after deformation. We have, therefore,

$$\frac{dL}{dp} = \alpha l\,(t^1 - t),$$

a condition which may be derived by putting the derivative with respect to p equal to zero of the expression,

$$L - \alpha l\,(t^1 - t)\,p = 0.$$

From the reasoning above, the vertical displacements of the crown joints for the two halves, as well as the rotations, must be equal and of opposite signs, so that we deduce as before,

$$\frac{dL}{ds} = 0,\ \frac{dL}{dm} = 0,$$

in addition to the previous equation, from which equations the three unknown quantities, p, s and m can be found as before. If it is desired to find the stresses due to temperature alone, then the terms in L pertaining to the loading should be omitted.

Solid Arch Hinged at Ends.

In this arch the position of the horizontal thrust p at the abutments is known, and by working from an abutment we can easily express $m_1, p_1, s_1, m_2, p_2, s_2, \ldots$, in terms of p alone, and thus find L from formula (10) of Section 12. For span invariable, to determine p, we put $\frac{dL}{dp} = 0$, and for a rise of temperature, $(t^1 - t)^\circ$, we have

$$\frac{dL}{dp} = dl\,(t^1 - t).$$

The last formula is evident, since if one end of the arch was free to slide horizontally the distance $\alpha l\,(t' - t)$ from the rise of temperature, the thrust p acting there would have to force it back a distance $\frac{dL}{dp}$ in the direction of the force, to cause the end to assume again its original position.

If one end of the arch is on rollers and a tie bar of length l, section ω and modulus e connects the ends, its increase under the horizontal thrust p, which equals the tension in it, is $\frac{lp}{e\omega}$. Here the end is moved in the opposite direction to the pull of the tie bar, so that if L is the work of the arch,

$$-\frac{dL}{dp} = \frac{lp}{e\omega} \therefore \frac{dL}{dp} + \frac{lp}{e\omega} = O.$$

from which p can be found.

There are no stresses due to temperature in this arch if the tie and arch are of the same material.

To get the vertical deflection of the crown under any loading, conceive a small force q to act vertically downward there, and find L the total work of deformation as before for any of the cases examined; then $\frac{dL}{dq}$ = vertical deflection at crown, no matter how small q is, and hence when $q = O$, for which the deflection is that due to the original loading.

Section 14.—Composite Structures.

Space forbids giving any extended numerical illustrations of the theory of least work; besides it is unnecessary, as Castigliano has devoted in his "Systemes Elastiques," 165 pages to the applications, which there should be no difficulty in following after what has preceded. Some of the numerical results, however, are interesting by way of comparison with approximate methods.

Thus in the trussed beam, Fig. 8, where the beam is of wood 8 meters long, $\overset{m}{0}.2 + \overset{m}{0}.2$ cross-section, and the ties of iron .024 meter in diameter, the load is taken at 180 kilograms per meter of length, and it is supposed that the truss is put together without stress at 15 degrees C. (59 degrees Fahr.).

The work of the two short wooden struts is neglected, because it is very small on account of their large section. The work of the longitudinal stress in the ties and beam is easily expressed by (3) of Section I, in terms of T, the tension in the inclined ties, putting the co-efficient of elasticity of iron as twelve times that of the wood. The work due to flexure of any one of the three segments of the horizontal beam can be computed by the formula $\frac{1}{2EI}\int M^2 dx$, by expressing M as a function of the distance x from one end of the segment (thus for end segment $M = -Rx + Tx \sin.\ \beta + \frac{1}{2}px^2$); or we can use, from (5) of Section 12, the formula,

$$\frac{1}{2EI}\frac{l}{3}(M_0^2 + 4M_1^2 + M_2^2),$$

where M_0 is the moment at one end, M_2 at the other and M_1 at the middle of the segment considered, or preferably a shorter formula that may be derived from this.

The sum of the elastic work of the ties and beam will now be found to be expressed in terms of T^2 and T, whence by Section 9 we find the true value of the tension T in the inclined ties, by placing the derivative of this sum in relation to T equal to zero, giving $T = 2\,120$ kilograms, from which the other stresses can at once be found.

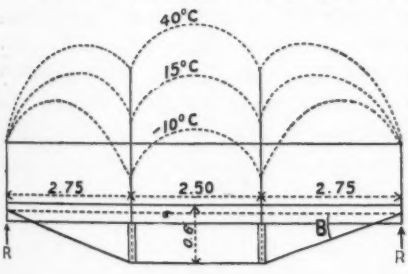

Fig. 8

To find the stresses at another temperature, by Section 6, we must add the term $-2\lambda T$ to the work already found, representing by λ the amount by which each inclined tie is too short when the horizontal tie is placed in position at the new temperature, and again place the derivative of the sum with respect to T equal to zero.

In this way we find at the temperature 40 degrees C. (104 degrees Fahr.), T = 1 380 kilograms, and at — 10 degrees C. (14 degrees Fahr.), $T = 2\,410$ kilograms. The bending moments are represented by the ordinates to the dotted curves shown in the figure, the maxima for 15 degrees C. and 40 degrees C. occurring at the center of the beam,

where the greatest compressive stress due to flexure and the uniform longitudinal compression combined, is found to be 20 kilograms per square centimeter at 15 degrees C., and 32 kilograms at 40 degrees C. The maximum stress at — 10 degrees C. occurs at 1.15 meters from the end and is 15 kilograms per square centimeter.

As it will be instructive to compare these exact results with approximate ones computed on the supposition of hinged joints, and that the beam does not aid by flexure except in carrying the load on any segment of it to the ends of that segment, I have computed for the end reaction 472 kilograms, $T = 2\,215$ kilograms, and the uniform compression along the beam $= 2\,163$ kilograms. Now, regarding the end-segment of the beam as a supported beam, 2.75 meters long and loaded with 180 kilograms per meter of length, we have the maximum compression in the top fiber $= 127\,630$ kilograms per square meter, which, added to the uniform compressive unit stress $2\,163 \div .04 = 54\,075$, gives a total maximum of 18.1705 kilograms per square centimeter. Similarly, the total maximum for the middle segment, regarded as "fixed at the ends," is 12.4385 kilograms per square centimeter—values below the 20 kilograms per square centimeter found above for 15 degrees C., and far below the 32 kilograms found for the 40 degrees C. We see from this comparison that whilst the values of T do not vary greatly for changes in temperature, yet the fiber stress in the beam experiences very wide changes for this combination of wood and iron, and that an approximate computation, as above, would not be on the side of safety, at least for the cross-sections given.

Curved Roof Truss with Tie Rods. Plate XXV.

In the elegant design shown in the plate, of a curved iron truss with tie rods, it will be found for assumed dimensions given in meters,* that for a symmetrical load of 760 kilograms per meter in length of roof, from Sections 0 to 2, and 864 kilograms from Sections 2 to 4, that the stresses in the ties marked T_1 and T_4 are respectively $T_1 = 16\,690$ and $T_4 = 2\,090$ kilograms. To find these values the elastic work of half the truss was found in terms of T_1 and T_4, and then the partial derivatives of this work

* The arched member has a total depth of 0.6 meter, and its flanges are each formed of similar irons, whose combined area $= .003898$ square meter, and moment of inertia, about a horizontal axis midway between the flanges, $= .000315$.

with respect to T_1 and T_4 put equal to zero. The two equations resulting give the above values of T_1 and T_4.

The usual statical computation on the supposition of hinged joints gives $T_1 = 18\,000$ and $T_4 = 1\,340$ nearly. There is a wide variation in the bending moments for cross-sections 0, 1, 2, 3, 4, as shown in the following table.

0	1	2	3	4
−318 −180	+110 +2223	−3489 0	−1313 +164	+1122 0

The upper figures refer to the exact, and the lower to the approximate, method of computation. The greatest fiber stress, including the longitudinal compression, is by the exact method, 807 at Section 2, and by the approximate, 770 kilograms per square centimeter at Section 1; so that the arch of uniform section would be safe if designed by the approximate method. Is it not best in practice, however, to actually make the joints at 0, 2 and 4 hinged, so that the sections can be designed by the ordinary laws of statics to carry any given stress? We can thus fit the sections exactly to the stresses and the influence of misfits does not come in to disturb the computations.

It would take us too far to enter further into the applications of the method of least work. It may be said in brief, as to the method to follow for any elastic system, whether eccentrically loaded or not, that the work of the members can generally be expressed in terms of the unknown stresses in certain bars; and that the partial derivatives with respect to these stresses, put equal to zero, furnish always as many equations as unknown stresses, so that the latter can readily be found by elimination.

It may be interesting to know that if the arched member above is replaced by a straight beam, as in the ordinary form of roof truss, that the greatest fiber stresses, as computed by the exact and approximate methods, do not differ much. Castigliano has also worked out the stresses in a crescent roof-truss with triangular bracing, connected to the arch member by pins passing through the centers of gravity of the cross-sections. The centers of pressure at these sections were frequently found

to lie as much as one-fourth the depth of the arch from the center, so that the fiber stresses were much larger than statics alone would give on the supposition of hinged joints; from which it may reasonably be inferred that the unit stresses in the continuous upper chords of our ordinary trusses are in excess of those given by usual methods, and simply because the hypothesis of joints perfectly free to move is never realized. It seems useless then to enter into any great refinement in our ordinary statical computations.

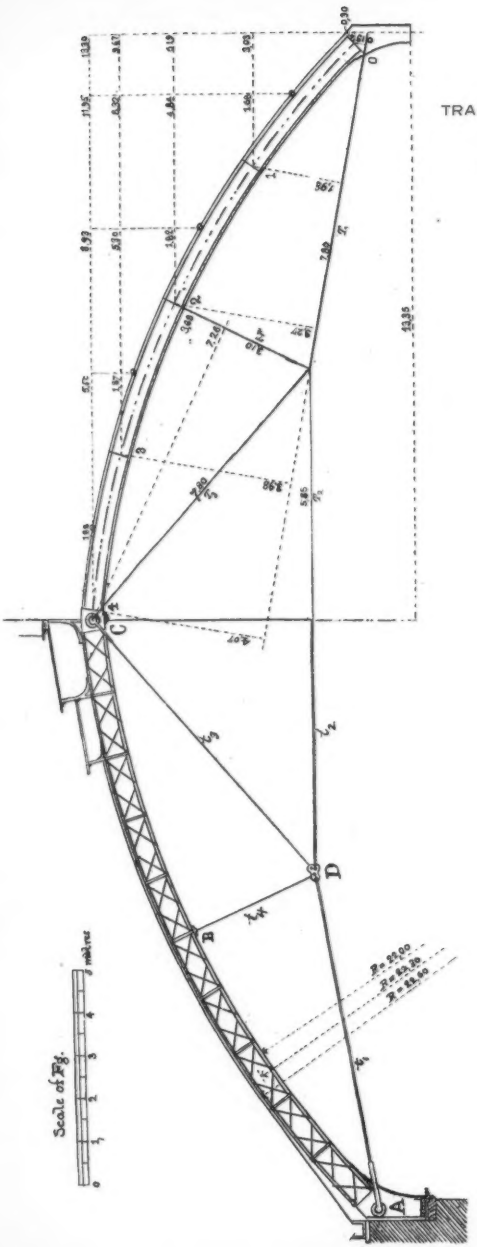

PLATE XXV.
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